Bicycles - A Mechatronics View

K. J. Åström
Lund University

1. Introduction
2. Modeling
3. Stabilization
4. Rear wheel steering
5. Maneuvering and stabilization
6. Conclusions
But don’t forget Control!
The US Army Motor Bike - A Design Disaster


“Many people have seen theoretical advantages in the fact that front-drive, rear-steered recumbent bicycles would have simpler transmissions than rear-driven recumbents and could have the center of mass nearer the front wheel than the rear. The U.S. Department of Transportation commissioned the construction of a safe motorcycle with this configuration. It turned out to be safe in an unexpected way: No one could ride it.”

The Santa Barbara Connection
Why Bicycles are Interesting?

• Very common efficient means of transportation and recreation
• Interesting non-trivial behavior
• Suitable to illustrate many issues in mechatronics
  – Modeling
  – Dynamics and control
  – Integrated systems and control design
  – The role of sensors and actuators
• Useful for education
  – Simple and illustrative experiments
  – High student attraction
Some Interesting Questions

• How do you stabilize a bicycle?
  – By steering or by leaning?
• Do you normally stabilize a bicycle when you ride it?
• Why is it possible to ride without touching the handle bar?
• How is stabilization influenced by the design of the bike?
• Why does the front fork look the way it does?
• The main message:
  – A bicycle is a feedback system, the action of the front fork is the key!
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Bicycle Modeling


“A bicycle is a doubly non-holonomic system; it has five degrees of freedom in finite motion, but only three such degrees in infinitesimal motion (rotation of the rear wheel in its instantaneous plane, to which the rotation of the front wheel is coupled by the condition of pure rolling; rotation about the handle bar axis; and common rotation of front and rear wheel about the line connecting their points of contact with the ground), as long as we do not consider the degrees of freedom of the cyclist himself.”
Arnold Sommerfeld on Gyroscopic Effects

That the gyroscopic effects of the wheels are very small compared with these (centripetal forces) can be seen from the construction of the wheel: if one wanted to strengthen the gyroscopic effects, one should provide the wheels with heavy rims and tires instead of making them as light as possible. It can nevertheless be shown that these weak effects contribute their share to the stability of the system.
Bicycle Modeling

Whitt and Wilson Bicycling Science MIT Press:

“The scientific literature (Timoshenko, Young, DenHartog et al) shows often complete disagreement even about fundamentals. One advocates that a high center of mass improves stability, another concludes that a low center of mass is desirable.”


More references later

Control theory gives very nice insights!
Coordinates and Variables
Major Simplifications

- Neglect elasticities.
- Five subsystems: frame, wheels, rider and front fork.
- Configuration space has dimension 8: position of rear wheel $x$ and $y$, orientation $\psi$, roll of frame and front wheel $\varphi$ and $\varphi_f$, steering angle $\delta$, wheel angles $\gamma_r$ and $\gamma_f$.
- Four non-holonomic constraints reduces degrees of freedom to 3.
- Assume constant forward velocity (-1 dof).
- Assume that rider is fixed to the bicycle (-1 dof).
- Let steering angle $\delta$ be the control variable (-1 dof).
- Resulting model has one dof, roll angle $\varphi$.
- Assume small tilt angles $\Rightarrow$ linear model.
Tilt Dynamics

Linearized momentum balance around $\xi$-axis

$$J \frac{d^2 \phi}{dt^2} - D \frac{d \omega}{dt} = mgh\phi + hF, \quad \omega = \frac{V}{b} \delta, \quad F = \frac{mV^2}{b} \delta$$
Linearized Tilt Dynamics

\[ J \frac{d^2 \phi}{dt^2} - D \frac{d\omega}{dt} = mgh\phi + hF, \quad \omega = \frac{V}{b} \delta, \quad F = \frac{mV^2}{b} \delta \]

Substituting and rearranging gives

\[ J \frac{d^2 \phi}{dt^2} - mgh\phi = \frac{mhV^2}{b} \delta + \frac{D}{b} \frac{d\delta}{dt} \]

Transfer function: \( P(s) = \frac{DV s + mhV / D}{bJ s^2 - mgh / J} \)

- **Poles:** \( p = \pm \sqrt{mgh / J} \approx \pm \sqrt{g / h} \)
- **Zero:** \( z = -mhV / D \approx V / a \)
- **Gain:** \( k = \frac{amhV}{bJ} \approx \frac{aV}{bh} \)

Approximations \( J \approx mh^2, \ D \approx mah \)
Summary

- Steering angle is the control variable
- Two states: tilt and tilt rate
- Transfer function with one zero and two poles one in the RHP
- What is the model good for?
- The model cannot explain that it is possible to ride without touching the handle bar
- What assumptions should be changed?
- Teaching kids to bike (Don’t use stiff arms!)
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A Slightly More Complicated Model

• It is not a good idea to assume that the steering angle is the control variable!

• The driver influences the bicycle primarily by applying torques to the handle bar.

• The front fork creates a feedback because leaning creates a torque on the front fork.

• A simple experimental verification.

• Important to understand the basic mechanisms!

• Improved model
  – Two subsystems: frame and front fork
  – One control variable: Torque on front fork
  – Dynamic model for frame static model for front fork
Block Diagram of a Bicycle

Control variable: Handlebar torque $T$

Process variables: Steering angle $\delta$, tilt angle $\phi$

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Overview

- The model of the frame should describe how the tilt of the frame $\phi$ is influenced by the steering angle $\delta$ influences (already done)
  - Dynamics is essential
  - What is the time scale? ($\omega = \sqrt{mg/h/J}$)
  - Storage of angular momentum ($\phi$ and $d\phi/dt$)

- The model of the front fork should tell how the steering angle $\phi$ is influenced by the torque on the handle bar $T$ and the tilt angle $\phi$
  - A static model will be used since the front fork dynamics faster than tilt dynamics
  - Tire-road interaction and gyroscopic effects will be neglected
The Front Fork

The front fork has many interesting features that were developed over a long time. Its behavior is complicated by geometry, the trail, tire-road interaction and gyroscopic effects. We will describe it by a very simple static linear model.

With a positive trail the front wheel lines up with the velocity, caster camber. The trail also creates a torque that turns the front fork into the lean. This torque is opposed by the caster-camber action and by torques generated by tire road interaction. A simple model is

$$\delta = k_1 T - k_2 \varphi$$

Experimental verifications?
Block Diagram of a Bicycle

Handlebar torque $T$

Front fork

$T$

$\delta$

$\varphi$

$k_1$

$k_2$

$\sum$

$k(s+V/a)$

$s^2-mgh/J$

$-1$

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The Closed Loop System

Combining the equations for the frame and the front fork gives

\[ J \frac{d^2 \varphi}{dt^2} - mgh \varphi = \frac{DV}{b} \frac{d \delta}{dt} + \frac{mhV^2}{b} \delta \]

\[ \delta = -k_2 \varphi + k_1 T \]

we find that the closed loop system is described by

\[ J \frac{d^2 \varphi}{dt^2} + \frac{k_2 DV}{b} \frac{d \varphi}{dt} + mgh \left( \frac{k_2 V^2}{bg} - 1 \right) \varphi = \frac{k_1 DV}{b} \left( \frac{dT}{dt} + \frac{mhV}{D} T \right) \]

This equation is stable if

\[ V > V_c = \sqrt{bg/k_2} \]

where \( V_c \) is the critical velocity. Physical interpretation. Think about this next time you bike!
Gyroscopic Effects

Gyroscopic effects has a little influence on the front fork

\[ J \frac{d^2 \phi}{dt^2} - mgh \phi + \frac{DV}{b} \frac{d\delta}{dt} + \frac{mhV^2}{b} \delta \]

\[ \delta = -k_2 \phi - k_g \frac{d\phi}{dt} + k_1 T \]

we find that the closed loop system is described by

\[ \left( J + \frac{k_g DV}{b} \right) \frac{d^2 \phi}{dt^2} + \left( \frac{k_2 DV}{b} + \frac{mhk_g V^2}{bg} \right) \frac{d\phi}{dt} + mgh \left( \frac{k_2 V^2}{bg} - 1 \right) \phi \]

\[ = \frac{k_1 DV}{b} \left( \frac{dT}{dt} + \frac{mhV}{D} T \right) \]

Damping can be improved drastically, but the stability condition is the same as before

\[ V > V_c = \sqrt{bg/k_2} \]
Summary

Modeling and feedback analysis gives good insight into stabilization of bicycles. A model of second order captures the essence.

• Simple models
  – Stiff rider
  – Static front fork model

• Interesting conclusion
  – Bicycle is self stabilizing if velocity is larger than the critical velocity. Design of the front fork is the key!

• Model can be augmented in several ways
  – Better front fork model centrifugal forces ...
  – Tire road interaction
Major Drawback of Simple Model

Parameters $k_1$ and $k_2$ in front fork model $\delta = k_1 T - k_2 \varphi$ are velocity dependent. A static model gives

$$k_1 = \frac{b}{acmg \cos \lambda \left( \frac{V^2 \cos \lambda}{bg} - \sin \lambda \right)}, \quad k_2 = \frac{1}{\frac{V^2 \cos \lambda}{bg} - \sin \lambda}$$

Closed loop system

$$\left( \frac{V^2 \cos \lambda}{bg} - \sin \lambda \right) J \frac{d^2 \varphi}{dt^2} + \frac{DV \cos \lambda}{b} \frac{d\varphi}{dt} + \left( mgh \sin \lambda - \frac{acmg \cos \lambda}{b} \right) \varphi$$

$$= \frac{DV h}{acmg} \frac{dT}{dt} + \frac{V^2 h}{acg} T$$

Bifurcations when velocity changes are very different.
Root Loci

Empirical front fork model

Physical front fork model

Effects of additional dynamics
More Complicated Models

• Front fork dynamics gives a fourth order model
• Linearized models have been obtained by many techniques
  – Complicated error prone calculations
  – Computational tools highly desirable
• Other factors
  – Rider lean
  – Tire road interaction
• Nonlinear models and parameter dependencies
• Tools like Modelica are indispensable
• Expressions for critical velocity complicated
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Rear Wheel Steering


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The Santa Barbara Connection
The Tilt Equation for Rear Wheel Steering

\[ J \frac{d^2 \varphi}{dt^2} - mgh \varphi = \frac{mhV^2}{b} \delta - \frac{D}{b} \frac{d \delta}{dt} \]
Compare with Front Wheel Steering

Front wheel steering:

\[ J \frac{d^2 \phi}{dt^2} - mgh \phi = \frac{mHV^2}{b} \delta + \frac{D}{b} \frac{d\delta}{dt} \]

Rear wheel steering:

\[ J \frac{d^2 \phi}{dt^2} - mgh \phi = \frac{mHV^2}{b} \delta - \frac{D}{b} \frac{d\delta}{dt} \]
The Linearized Tilt Equation

The linearized equation becomes

\[ J \frac{d^2 \phi}{dt^2} - mgh \phi = \frac{mhV^2}{b} \delta - \frac{D}{b} \frac{d\delta}{dt} \]

The transfer function of the system is

\[ P(s) = \frac{DV}{bJ} \frac{-s + \frac{mhV}{D}}{s^2 - \frac{mgh}{J}} \]

One pole and one zero in the right half plane.

Physical interpretation.
Rear Wheel Steering

The transfer function from steering angle to tilt is

$$P(s) = \frac{DV}{bJ} s^2 + \frac{mhV}{D}$$

One RHP pole at $p = \sqrt{\frac{mgh}{J}} \approx 3 \text{ rad/s}$

One RHP zero at $z = \frac{mhV}{D}$

Pole position independent of velocity but zero proportional to velocity. When velocity increases from zero to high velocity you pass a region where $z = p$ and the system is uncontrollable. The system is impossible to control when $z = p$. 

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Does Feedback from Rear Fork Help?

Combining the equations for the frame and the rear fork gives

\[
J \frac{d^2 \phi}{dt^2} - mgh \phi - \frac{amhV}{b} d\delta \frac{d\phi}{dt} + \frac{mhV^2}{b} \delta
\]

\[
\delta = -k_2 \phi + k_1 T
\]

we find that the closed loop system is described by

\[
J \frac{d^2 \phi}{dt^2} - \frac{k_2}{b} \frac{d\phi}{dt} + mgh \left( \frac{k_2 V^2}{bg} - 1 \right) \phi = \frac{k_1 V}{b} \left( \frac{dT}{dt} + \frac{mhV}{D} T \right)
\]

where \( V_c = \sqrt{bg/k_2} \). This equation is unstable for all \( k_2 \).

There are several ways to turn the rear fork but it makes little difference.

Can the system be stabilized robustly with a more complex controller?
Limitations due to Non-minimum Phase Dynamics

Factor process transfer function as \( P(s) = P_{mp}(s)P_{nmp}(s) \) such that \(|P_{nmp}(i\omega)| = 1\) and \(P_{nmp}\) has negative phase. Requiring a phase margin \(\phi_m\) we get

\[
\arg L(i\omega_{gc}) = \arg P_{nmp}(i\omega_{gc}) + \arg P_{mp}(i\omega_{gc}) + \arg C(i\omega_{gc}) \\
\geq -\pi + \phi_m
\]

With \(P_{mp}(s)C(s) = (s/\omega_{gc})^n\) (Bode’s ideal loop transfer function) we get \(\arg P_{mp}C = n\pi/2\) and

\[
\arg P_{nmp}(i\omega_{gc}) \geq -\pi + \phi_m - n\frac{\pi}{2}
\]

This equation holds approximatively for other designs if \(n\) is the slope at the crossover frequency.
Bode Plots Should Look Like This

Gain curve

Phase curve

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Summary of Limitations - Part 1

• A RHP zero \( z \) gives an upper bound to bandwidth

\[
\frac{\omega_{gc}}{z} \leq \begin{cases} 
0.5 & \text{for } M_s, M_t < 2 \\
0.2 & \text{for } M_s, M_t < 1.4.
\end{cases}
\]

Slow RHP zeros difficult! Why \( e^{-sT} \approx \frac{1-sT/2}{1+sT/2} \), \( s = 1/2T \)

• A RHP pole \( p \) gives a lower bound to bandwidth

\[
\frac{\omega_{gc}}{p} \geq \begin{cases} 
2 & \text{for } M_s, M_t < 2 \\
5 & \text{for } M_s, M_t < 1.4.
\end{cases}
\]

Fast RHP poles difficult!

• RHP poles and zeros must be sufficiently separated

\[
\frac{z}{p} \geq \begin{cases} 
6.5 & \text{for } M_s, M_t < 2 \\
14.4 & \text{for } M_s, M_t < 1.4.
\end{cases}
\]
System with RHP Pole and Zero Pair

\[ P_{nmp}(s) = \frac{(z - s)(s + p)}{(z + s)(s - p)} \]

For \( z > p \) the cross over frequency inequality becomes

\[ \frac{\omega_{gc}}{z} + \frac{p}{\omega_{gc}} \leq (1 - \frac{p}{z}) \tan\left(\frac{\pi}{2} - \frac{\phi_m}{2} + n_{gc} \frac{\pi}{4}\right) \]

\[ \phi_m < \pi + n_{gc} \frac{\pi}{2} - 2 \arctan \frac{\sqrt{p/z}}{1 - p/z} \]

With \( n_{gc} = -0.5 \) we get

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<th>5</th>
<th>5.83</th>
<th>8.68</th>
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Systems with RHP poles and zeros can be controlled robustly only if poles and zeros are well separated.

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The zero-pole ratio is

\[
\frac{z}{p} = \frac{V}{D} \sqrt{\frac{mhJ}{g}} \approx \frac{V}{a} \sqrt{\frac{h}{g}}
\]

The system is difficult to control robustly if this ratio is greater than 6.

To make the ratio large you can

- Make \( \alpha \) small by leaning forward
- Make \( V \) large by biking fast (takes guts)
- Make \( h \) large by standing upright
- Move back and sit down when the velocity is sufficiently large
Richard Klein and His Bikes
Bicycles in Education

- Not High Tech but all students are familiar with them
- Good illustration of many interesting issues in control
- Summer course for European student in Lund 1996
- Regular use in introductory courses at Lund and UCSB, Hannover 2000 (Lunze), Michigan (Stefanopoulou)
- How bicycles are used?
  - Motivation
  - Concrete illustration of ideas and concepts
  - Experiments
- Very high student attraction
The Santa Barbara experience

- Why should a mechanical engineer learn control
- Argue the design of a recliner
- Demo
- Motivate limitations and why it is worth while to know about poles and zeros
- Student feedback
  - Control theory really gives insight
  - I did not know that a bike could be self stabilizing
  - Now I know why RHP poles and zeros are important
  - Even if I will not specialize in control I have a feel for the questions I should ask if I became involved with a control system
The UCSB Bike
The Hamburg-Bochum Bike
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Maneuvering and Stabilization - A Classic Problem

Lecture by Wilbur Wright 1901:

Men know how to construct airplanes.
Men also know how to build engines.
Inability to *balance and steer* still confronts students of the flying problem.
When this one feature has been worked out, the age of flying will have arrived, for all other difficulties are of minor importance.

The Wright Brothers figured it out and flew the Kitty Hawk on December 17 1903!

Minorsky 1922:

It is an old adage that a stable ship is difficult to steer.
Maneuvering

Having understood stabilization of bicycles we will now investigate steering for the bicycle with a rigid rider.

• Key question: How is the path of the bicycle influenced by the handle bar torque?

• Steps in analysis, find the relations
  – How handle bar torque influences steering angle
  – How steering angle influences velocity
  – How velocity influences the path

We will find that the instability of the bicycle frame causes some difficulties in steering (dynamics with right half plane zeros). This has caused severe accidents for motor bikes.
Block Diagram Analysis Useful

Transfer function from $T$ to $\delta$ is

$$\frac{k_1}{1 + k_2 P(s)} = \frac{k_1}{1 + k_2 \frac{k(s+V/a)}{s^2 - mgh/J}} = k_1 \frac{s^2 - mgh/J}{s^2 + \frac{amhk_2 V}{bJ} s + \frac{mgh}{J} \left( \frac{V^2}{V_c^2} - 1 \right)}$$
Steering Angle to Path

\[
\begin{align*}
\frac{dy}{dt} &= V\psi \\
\frac{d\psi}{dt} &= \frac{V}{b}\delta \\
L\delta &= G_{\delta T}L T \\
G_{\delta T} &= \frac{k_2}{1 + k_1G_{\phi\delta}(s)}
\end{align*}
\]

Transfer function from steering torque to path deviation

\[
G_{yT} = \frac{k_1 V^2}{b(1 + k_1G_{\phi\delta}(s))} = \frac{k_1 V^2 \left(s^2 - \frac{mgh}{J}\right)}{bs^2 \left(s^2 + \frac{amhk_1V_0}{bJ}s + \frac{mgh}{J} \left(\frac{V_0^2}{V_c^2} - 1\right)\right)}
\]
Assume that the bicycle is driven along a straight line and that a constant torque is applied to the handle bar. What path will the bicycle take?
Summary

• The simple second order model with a rigid rider can explain stabilization. The model indicates that steering is difficult due to the right half plane zero in the transfer function from handle bar torque to steering angle.

• The right half plane zero has some unexpected consequences which gives the bicycle a counterintuitive behavior. This has caused many motorcycle accidents.

• How can we reconcile the difficulties with our practical experience that a bicycle is easy to steer?

• The phenomena depends strongly on the assumption that the rider does not lean.

• The difficulties can be avoided by introducing an extra control variable (leaning).
Coordinated Steering

An experienced rider uses both lean and the torque on the handle bar for steering. Intuitively it is done as follows:

- The bicycle is driven so fast so that it is automatically stabilized.
- The turn is initiated by a torque on the handle bar, the rider then leans gently into the turn to counteract the centripetal force which will tend to lean the bike in the wrong direction. This is particularly important for motor bikes which are much heavier than the rider.

A proper analysis of a bicycle where the rider leans require a more complex model because we have to account for two bodies instead of one. There are also two inputs to deal with. Accurate modeling of a bicycle also has to consider tire road interaction and a more detailed account of the mechanics.
Sensors, Actuators Poles and Zeros

• Watch our for time delays, and RHP poles and zeros
• Zeros can be eliminated by adding sensors and actuators!
The zeros of the standard linear system \((A, B, C)\) are the values of \(s\) where the matrix

\[
\begin{pmatrix}
  sI - A & B \\
  C & 0 \\
\end{pmatrix}
\]

looses rank
• Big difference between using observers and sensors as far as robustness are concerned!
• A naive observation on the rear steered bike

\[
J \frac{d^2 \phi}{dt^2} - \frac{k_2DV}{b} \frac{d\phi}{dt} + mgh \left( \frac{k_2V^2}{bg} - 1 \right) \phi = \frac{k_1DV}{b} \left( \frac{dT}{dt} + \frac{V}{aT} \right)
\]

Adding a rate gyro solves the problem!
Bicycles - A Mechatronics View

K. J. Åström

1. Introduction
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3. Stabilization
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5. Maneuvering and stabilization
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Conclusions

• Bicycles illustrate many issues in mechatronics
  – Modeling
  – Integrated system and control design
  – The role of sensors and actuators

• Bicycles are cheap and very useful in education
  – Stabilization and steering
  – Nice way to illustrate zeros

• Lesson 1: Dynamics is important! A system may look OK
  statically but un-tractable because of dynamics.

• Lesson 2: A system that is difficult to control because of
  zeros in the right half plane can be improved significantly
  by introducing more sensors and actuators.
References


