A hybrid control scheme for swing-up acrobatics*

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Abstract

This paper describes a hybrid control scheme to swing-up a pendulum mounted on a cart (cart-pole system) or on another pendulum (pendubot system). Switches between equilibria of the actuated link are used to swing-up the free link. It is argued that this hybrid design is robust and versatile, considerably simplifies the closed-loop analysis, and is well-suited for performance optimization.

1 Introduction

Swinging up a single pendulum is an elementary nonlinear control problem. Swinging up a pendulum mounted on a cart (cart-pole system) or on another pendulum (pendubot system) has proven feasible but not straightforward. Studies by many authors (e.g. [Spo97, SP96, Tee96, FLS00, LMR98, MP96]) have shown that these examples, as particular instances of two-degrees of freedom mechanical systems with one actuator, constitute challenges both for stabilization (or energy-based) methods and path-planning (or inversion-based) methods.

The objective pursued in this paper is to illustrate on these by now classical nonlinear control problems the versatility of hybrid control architectures where the design is conceived as a library of elementary pieces of trajectories plus an automaton in charge of ordering (in time) the right sequence of such 'primitives'. We show that the simplicity of the single pendulum control design can be retained for the two-degrees of freedom problem with, in addition, a straightforward adaptation from the cart-pole situation to the pendubot situation. This is in contrast with several earlier designs which are more dependent on the details of the system dynamics. At the same time, our design is not based on heuristics and no nonlinearities are neglected, so that convergence of the closed-loop trajectories to the desired equilibrium can be rigorously proven.

The illustration in this paper is extremely simple but the underlying design principle is of considerable generality, especially for complex robotic applications. It has obvious robustness with respect to model uncertainties because the control laws do not depend too much on the detailed dynamics. It is also very well suited to performance optimization because the untractable task of solving a nonlinear continous-time optimization problem can be replaced by a tractable discrete optimization of the automaton. This has been discussed in the recent paper [FDF99]. The present note includes experimental results with a pendubot system but the controller optimization and its learning capabilities will be the object of future work.

The paper is organized as follows. Section 2 describes the basic swing-up control problem of a single pendulum, together with the path planning solution and the stabilization solution. In Section 3, we briefly explain the reasons why the extension of these designs to the two-degrees of freedom problem is not so straightforward and why our proposed hybrid control scheme trivially solves the problem. This design is adapted in Section 4 to the pendubot situation, both to illustrate the versatility of the approach and because the strategy has been experimentally tested on a pendubot. Concluding remarks are presented in Section 5.

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2 Swinging up a single pendulum

Swinging up a single pendulum is easy. Denote by J the moment of inertia with respect to the pivot point, m the mass, τ the applied torque, θ the angle between the vertical and the pendulum, and $\omega = \dot{\theta}$ the angular velocity. The equations of motion are given by

$$J\dot{\omega} = mgl\sin\theta + \tau \tag{1}$$

This second-order model is invertible (or feedback linearizable), that is, the feedback transformation $\tau = -mgl\sin\theta + u$ transforms the nonlinear system (1) into the linear system $J\dot{\omega} = u$. Defining the tracking error

$$e = \theta - \theta^*,$$

any desired trajectory $(\theta^*(t), \omega^*(t))$ can be implemented through the PD-control law

$$u = J\dot{\omega}^* - K_P e - K_D \dot{e},\ K_P > 0, K_D > 0$$

To swing-up the pendulum, it suffices to design a trajectory that connects the stable equilibrium $(\theta, \omega) = (-\pi, 0)$ to the unstable equilibrium $(\theta, \omega) = (0, 0)$. For instance, the trajectory

$$\theta^*(t) = \begin{cases} \frac{1}{2}ct^2 - \pi, & 0 \le t \le \frac{T}{2} \\ -\frac{1}{2}c(t-T)^2, & \frac{T}{2} \le t \le T \end{cases}$$

with $c = \frac{4\pi}{T^2}$ results in swing-up in time T with a bang-bang acceleration $\dot{\omega}^*$ of amplitude $c = \frac{4\pi}{T^2}$.

This swing-up strategy is known as the "computed torque" in the robotics literature. It is a "path planning" approach, which requires a good model and strong invertibility properties but works very well under these assumptions [SL91].

There is an alternative strategy to swing up the pendulum, known as energy-based or Lyapunov-based approach [AF96]. The energy of the (free) pendulum is given by

$$E = mgl(\cos\theta - 1) + \frac{1}{2}J\omega^2 \tag{2}$$

where the constant has been selected so that the energy is maximum and zero in the upright equilibrium position. Lyapunov-based control of the pendulum relies on the property that the time-derivative of the energy can be assigned by the control torque:

$$\dot{E} = \omega\tau \tag{3}$$

Using the feedback control

$$\tau = -\mathrm{sat}(E)\mathrm{sign}(\omega) \tag{4}$$

results in the closed-loop system

$$\dot{E} = -|\omega|\mathrm{sat}(E)$$

The energy along the closed-loop solutions converges to zero, which is the energy level of the homoclinic orbit through the unstable equilibrium. A local stabilizing control is then able to catch the pendulum in the inverted position. The details of the control law (4) are not important, provided energy is pumped into the system in the average. For this reason, energy-based control strategies are quite robust. Also the invertibility requirements of path-planning methods are relaxed with energy-based control strategies. For instance, with a view on the cart-pole example treated in the next section, consider the situation where the torque control in (1) is replaced by a lateral acceleration control a of the pivot:

$$J\dot{\omega} = mgl\sin\theta - ml\cos\theta\ a \tag{5}$$

In this case, the feedback linearizing control $a = -\frac{1}{\cos\theta}(g\sin\theta + u)$ faces a singularity at $\theta = \pi/2$. Instead, the energy-based acceleration control

$$a = \mathrm{sat}(E)\mathrm{sign}(\omega\cos\theta) \tag{6}$$

ensures an increase of energy along closed-loop solutions and the analysis is essentially unchanged with respect to the torque control situation.

3 Swing-up acrobatics for the cart-pole system

3.1 Adding the cart dynamics to the pendulum equation

Denoting by x the cart position, $v = \dot{x}$ the cart velocity, and F the lateral force applied to the cart (Figure 1), the cart-pole equations of motion are

$$\begin{array}{rcl} J\dot{\omega} - mgl\sin\theta + ml\dot{v}\cos\theta & = & 0 \\ M\dot{v} + ml\dot{\omega}\cos\theta - ml\omega^2\sin\theta & = & F \end{array} \tag{7}$$

or equivalently

$$\begin{array}{rcl} J\dot{\omega} - mgl\sin\theta + ml\dot{v}\cos\theta & = & 0 \\ (M - m\cos^2\theta)\dot{v} - ml\omega^2\sin\theta + mg\sin\theta\cos\theta & = & F \end{array}$$

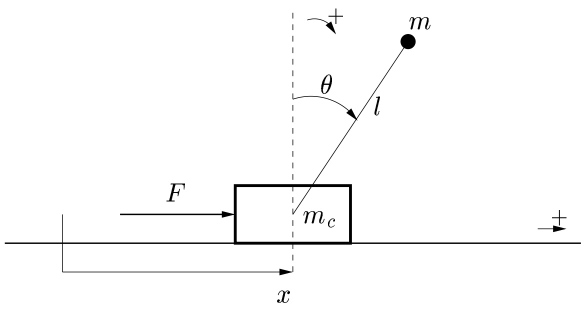

Figure 1: The cart-pole system.

where $J = ml^2$ and $M = m + m_c$.

The (nonsingular) feedback transformation

$$(M - m\cos^2\theta)a - ml\omega^2 \sin\theta + mg\sin\theta\cos\theta = F$$

yields the simplified dynamics

$$\begin{aligned} J\dot{\omega} - mgl\sin\theta + mla\cos\theta &= 0 \qquad (8) \\ \dot{v} &= a \qquad (9) \end{aligned}$$

where the new input a directly controls the cart acceleration. Note that (8-9) is simply the single pendulum model (5) augmented with the cart dynamics.

Contrary to the single pendulum system, the cart-pole system is no longer invertible. Approximate path planning based on the Jacobian linearization of (8-9) will be effective only for trajectories that maintain $\theta \approx const$, but not for swing-up acrobatics.

Extending the Lyapunov-based approach from the single pendulum to the cart-pole system is not straightforward either. The energy (2) of the pendulum and the kinetic energy of the cart can be taken into account to yield for example

$$W = E^2 + v^2$$

and the energy-based acceleration control

$$a = -\mathrm{sat}(W)\mathrm{sign}(-ml\omega E\cos\theta + v)$$

This control law will swing-up the pendulum and asymptotically stop the cart but is has no authority on the asymptotic position of the cart. Various methods have been proposed in the literature to add 'integral action' to this control law in such a way that asymptotic stability of the desired equilibrium is guaranteed with a large basin of attraction (e.g. [Tee96, Pra00, JSK96]). These design methodologies have been called *forwarding* stabilization methods because they handle augmented dynamics that result from feedforward connections only in a bloc-diagram representation of the system. This is in contrast with *backstepping* stabilization methods which handle augmented dynamics that result from feedback connections only in a bloc-diagram representation of the system (see [SJK97] for a detailed comparison of these stabilization methods).

3.2 Shaking the cart to swing-up the pole

Consider the problem of steering the cart-pole system (8-9) from the $(x = 0, \theta = \pi)$ equilibrium to the $(x = 0, \theta = 0)$ equilibrium. We propose to swing-up the pole by an energy-based approach while making sure that the cart stays sufficently close to the $x = \dot{x} = 0$ state. To this end, we suggest to select two "attractors" at $x = x_- = -\Delta/2$ and $x = x_+ = \Delta/2$, and to switch between these two attractors in such a way that the pendulum energy

$$E = mgl(\cos\theta - 1) + \frac{1}{2}J\omega^2 \qquad (10)$$

increases "in the average". The special choice of the cart trajectories (switch between equilibria) makes it easy to control the maximal cart excursion (a consideration of importance for practical implementation) and trivially solves the problem of stabilizing the cart position (the control parameter Δ can be chosen to decrease to zero as the pendulum approaches its inverted position or a final linear correction can be implemented to transfer the equilibrium from $(x = \pm\Delta, \theta = 0)$ to $(x = 0, \theta = 0)$). At the same time, an appropriate switching between these elementary cart trajectories will realize the swing-up of the pendulum. This is because the details of the energy pumping are unimportant.

The switching scheme for the cart is now selected to be most efficient in increasing the energy of the pole. From (8) and (10), one has

$$\begin{aligned} \dot{E} &= J\omega\dot{\omega} - mgl\omega\sin\theta \\ &= -mla\,\omega\cos\theta, \qquad (11) \end{aligned}$$

which shows that the acceleration a of the cart is most efficient for increasing E when the pole crosses the lower vertical position. This calls for the following strategy: starting near the attractor x_- (i.e. $x \simeq x_-$), wait for the pole to cross the lower vertical (i.e. $\theta \simeq \pi$) from right to left (i.e. $\omega \geq 0$), then give a short and violent acceleration $a = a_{max}$ and decelerate smoothly while the cart approaches the other attractor x_+. This guarantees that E increases during the transfer from x_- to x_+. The complete strategy consists in realizing a succession of such jerky transfers between x_- and x_+.

The simplicity of the above control architecture allows for many possible refinements and a further performance analysis. For instance, the elegant analysis in [AF96] relating the maximal available acceleration $a_{\max}$ to the number of swings necessary to achieve the swing-up directly applies to the present situation. The transfer dynamics between the cart equilibria x_- and x_+ can also be optimized to smooth out the acceleration switches without too much degradation of the energy pumping.

4 Adaptation of the control law to the pendubot system

The pendubot is conceptually very close to the cart-pole system. The translational degree of freedom of the cart pole is replaced by a rotational degree of freedom. This creates an additional centrifugal term in the (free) pendulum dynamics and the details of the overall dynamics are somewhat different. However, a versatile control architecture should not vary too much from the cart-pole to the pendubot. We show in this section that our proposed design is virtually unchanged when applied to the pendubot.

4.1 Equations

The Pendubot [SB95] is a two-link planar robot moving in a vertical plane with an actuator at the fixed shoulder and no actuator at the elbow (Figure 2). The equations of motion are

$$M\ddot{q} + h(q,\dot{q}) + \phi(q) = F \tag{12}$$

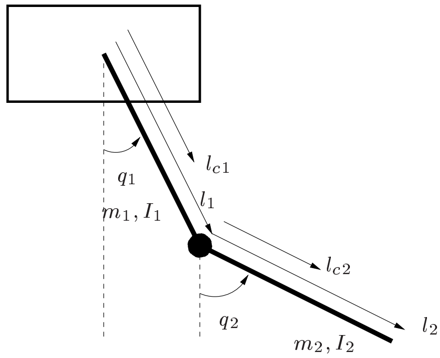

Figure 2: The Pendubot

where

$$\begin{aligned}
q &= (q_1, q_2)^T \\
F &= (\tau, 0)^T \\
\phi(q) &= (P_4 g \sin(q_1), P_5 g \sin(q_2))^T \\
h(q,\dot{q}) &= (-P_3 \sin(q_2 - q_1)\,\dot{q}_2^2, P_3 \sin(q_2 - q_1)\,\dot{q}_1^2)^T \\
M &= \begin{bmatrix} P_1 & P_3\cos(q_2 - q_1) \\ P_3\cos(q_2 - q_1) & P_2 \end{bmatrix} \\
P_1 &= m_1 l_{c1}^2 + m_2 l_1^2 + I_1 \\
P_2 &= m_2 l_{c2}^2 + I_2 \\
P_3 &= m_2 l_1 l_{c2} \\
P_4 &= m_1 l_{c1} + m_2 l_1 \\
P_5 &= m_2 l_{c2}.
\end{aligned}$$

The feedback transformation

$$\begin{aligned}
\tau = {} & (M_{11} - M_{12} M_{22}^{-1} M_{21})\, a \\
& + h_1 - M_{12} M_{22}^{-1} h_2 \\
& + \phi_1 - M_{12} M_{22}^{-1} \phi_2
\end{aligned} \tag{13}$$

yields the simplified dynamics

$$\begin{aligned}
P_2\ddot{q}_2 + P_3\cos(q_2 - q_1)a \qquad & \\
+P_3\sin(q_2 - q_1)\dot{q}_1^2 + P_5 g \sin(q_2) &= 0 \qquad (14) \\
\ddot{q}_1 &= a. \qquad (15)
\end{aligned}$$

Note the similarity between (14) and (8) when q_1 is close to π and $\dot{q}_1$ is small.

In the absence of forcing term, the Pendubot admits four equilibrium positions: DD, DU, UD and UU (where DU means that the actuated link is in the vertical downward position and the free link is in the vertical upward position, etc.).

4.2 Control algorithm

The following control algorithm steers the Pendubot from the stable DD position to the unstable UU position. The algorithm decomposes into three phases.

Phase 1: from DD to UD.
The transfer is achieved by selecting a reference trajectory $q_{1r}(t)$ starting at D and leading up to U. This trajectory is stabilized by means of a PD controller on (15). Because the free link is not controlled during that phase, the transfer should not be too fast in order to avoid large oscillations of the free link. (Alternatively, the reference trajectory $(q_{1r}(t), \dot{q}_{1r}(t), 0, 0)$ can be very well stabilized on the basis of the Jacobian linearization).

Phase 2: from UD to UU by successive jerks.
This part of the algorithm applies the principle described in Section 3.1. It is not very important that Phase 2 starts exactly from UD. The switching criterion between q_{1+} and q_{1-} is the same as for the cart-pole system, and the transfer dynamics are simply chosen to result from a PD control on (15):

$$\ddot{q}_1 = -K_P(q_1 - q_1^{(\epsilon)}) - K_D\dot{q}_1$$

where $q_1^{(\epsilon)} = \epsilon\Delta/2$, $\epsilon \in \{+1, -1\}$. The parameters K_P, K_D and Δ are chosen so as to achieve critical damping of (15) and to bring E smoothly to 0.

Phase 3: stabilizing UU.
When the free link comes sufficiently close to the upper position, the control algorithm switches to a linear feedback law stabilizing the UU position.

4.3 Simulations

We simulated our algorithm with MATLAB and SIMULINK. Figure 3 corresponds to parts 2 and 3.

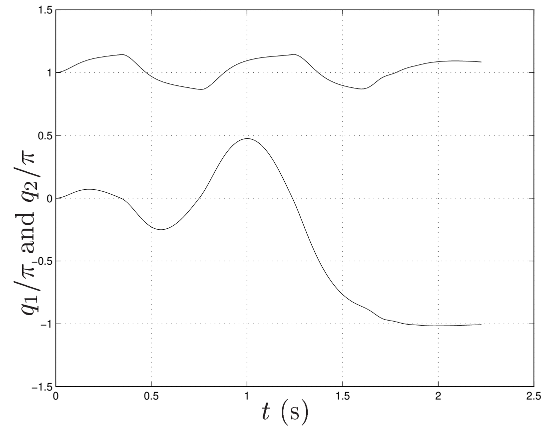

Figure 3: Simulation of phases 2 and 3 of the complete swing-up procedure for the Pendubot. The upper curve corresponds to q_1 and the lower to q_2. Switching between phases 2 and 3 occurs around $t = 1.5$. The parameters of the control law have been chosen such that 4 switchings occur during phase 2.

4.4 Experimental results

The algorithm has been applied with success to a real Pendubot in our control laboratory (Figure 4).

The robustness of the proposed design is illustrated by the success of the swing-up in spite of parameter uncertainty. We use the values from the manual [Mec98] for the parameters P_1 to P_5 but no identification has been performed in our lab; the estimated error is about 10% on each parameter. As a consequence, the feedback transformation (13) is unprecisely implemented and there is a discrepancy between the theoretical and actual trajectories of the actuated link, which is visible on the q_1 trace in Figure 4 (the discrepancies become larger as the oscillations of the free link increase because there is a larger uncertainty on P_2 and P_3). This does not affect the feasability of the swing-up strategy.

5 Conclusion

A swing-up strategy is designed for the cart-pole system that consists in switching appropriately between two prefered positions of the cart so as to pump energy in the pendulum. The design is 'exact' in the sense that no nonlinearities are neglected and convergence of closed-loop solutions can be analysed. At the same time, it does not very much depend on the details of the dynamics. As a consequence, the control architecture is robust, versatile, and well-suited for performance optimization.

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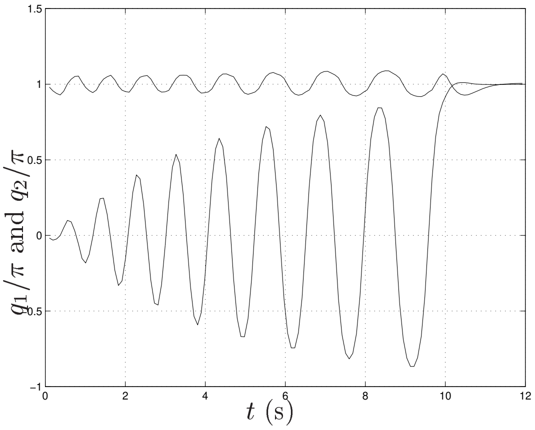

Figure 4: Experimental data for phases 3 and 4 of the complete swing-up procedure.

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