STABILITY OF PERTURBED DELAY DIFFERENTIAL EQUATIONS AND STABILIZATION OF NONLINEAR CASCADE SYSTEMS

W. MICHIELS †, R. SEPULCHRE ‡, AND D. ROOSE †

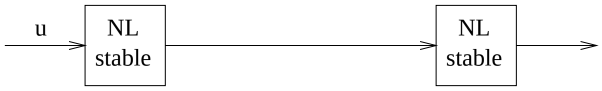
Abstract. In this paper the effect of bounded input perturbation on the stability of nonlinear globally asymptotically stable delay differential equations is analyzed. We investigate under which conditions global stability is preserved and if not, whether semi-global stabilization is possible by controlling the size or shape of the perturbation. This results in a general framework, in which the stabilization of partial linear cascade systems using partial state feedback can be treated systematically.

Key words. cascade systems, delay equations, nonlinear control

AMS subject classifications. 34K20,93A20,93D15

1. Introduction.

The stability analysis of the series (cascade) interconnection of two stable nonlinear systems described by ordinary differential equations is a classical subject in system theory [13, 14, 17].



Contrary to the linear case, the zero input global asymptotic stability of each subsystem does not imply the zero input global asymptotic stability of the interconnection. The output of the first subsystem acts as a transient input disturbance which can be sufficient to destabilize the second subsystem. In the ODE case, such destabilizing mechanisms are well understood, since the seminal work by Sussmann and Kokotovic [15]. They can be subtle but are almost invariably associated to a finite escape time in the second subsystem (Some states blow up to infinity in a finite time). The present paper explores similar instability mechanisms generated by the series interconnection of nonlinear DDEs. In particular we consider the situation where the destabilizing effect of the interconnection is delayed and examine the difference with the ODE situation.

Instrumental to the stability analysis of cascades, we first study the effect of external (affine) perturbations w on the stability of nonlinear time delay systems

$$\dot{z} = f(z, z(t-\tau)) + \Psi(z, z(t-\tau))w, \quad z \in \mathbb{R}^n,\ w \in \mathbb{R}, \tag{1.1}$$

where we assume that the equilibrum $z = 0$ of $\dot{z} = f(z, z(t-\tau))$ is globally asymptotically stable. We consider perturbations $w = \eta(t)$ which belong to both L_1 and L_∞ and investigate the region in the space of initial conditions which give rise to bounded solutions under various assumptions on the system and the perturbation. These results are strenghthened to asymptotic stability results when the perturbation is generated by a globally asymptotically stable ODE.

We consider both global and semi-global results. In the ODE-case, an obstruction to global stability is formed by the fact that arbitrarily small input perturbations

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can cause the state to escape to infinity in a finite time, for instance when the interconnection term $\Psi(z)$ is nonlinear in z. This is studied extensively in the literature in the context of stability of cascades, see e.g. [15] [13] and the references therein. Even though delayed perturbations do not cause a finite escape time we explain a similar mechanism giving rise to unbounded solutions, caused by nonlinear delayed interconnection terms.

In situations where unbounded solutions are inevitable for large initial conditions, we investigate under which conditions trajectories can be bounded semi-globally in the space of initial conditions, in case the perturbation is parametrized, i.e. $\eta = \eta(t, a)$. Hereby we let the parameter a control the L_1 or L_∞ norm of the perturbation. We also consider the effect of concentrating the perturbation in an arbitrarily small time-interval. The study of controlled perturbations is motivated by the situation where the perturbation is the output of a controlled system, see Figure 1.1.

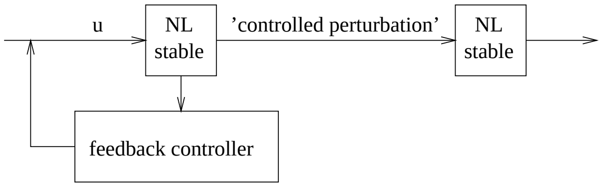

FIG. 1.1. *Partial state feedback as a way of controlling the input perturbation to the second subsystem*

In the second part of the paper, we assume that the perturbation to (1.1) is generated by a controlled linear system and study the stabilization of the cascade,

$$\left\{ \begin{array}{l} \dot{z} = f(z, z(t-\tau)) + \Psi(z, z(t-\tau))y \\ \dot{\xi} = A\xi + Bu \\ y = C\xi, \qquad \xi \in \mathbb{R}^\mu, \quad u, y \in \mathbb{R} \end{array} \right. \tag{1.2}$$

with the pair (A, B) controllable, using a feedback law of the form

$$u = F\xi. \tag{1.3}$$

In the ODE-case this stabilization problem has been extensively studied in the literature, for instance in [16][1][15][8]. Because the output of the linear subsystem, which acts as a destabilizing disturbance to the nonlinear subsystem, can cause trajectories to escape to infinity in a finite time, one typically tries to drive the 'perturbation' y quickly to zero. However, a high-gain control, placing all observable eigenvalues far into the left half plane, will not necessarily result in large stability regions, because of the fast peaking phenomenon [15] [13]. Peaking is a structural property of the ξ-subsystem whereby achieving faster convergence implies larger overshoots which can in turn destabilize the cascade. Semi-global stability results are obtained when imposing structural assumptions on the ξ-subsystem (a nonpeaking system) or by imposing conditions on the z-subsystem and the interconnection term Ψ: for instance in [13, Theorem 4.41] one requires a nonpeaking linear susbsysem and the conditions of [15, Theorem 9,1] are a trade-off between peaking and growth.

The structure of the paper is as follows. After some preliminaries (Section 2), we study the effect of bounded input perturbations in Sections 3 and 4 and use the obtained results to study the stabilization of partial linear cascades with partial state feedback in Section 5.

2. Preliminaries. The state of the delay equation (1.1) at time t can be described as a vector $z(t) \in \mathbb{R}^n$ or as a function segment z_t defined by

$$z_t(\theta) = z(t+\theta),\ \theta \in [-\tau,\ 0].$$

Therefore delay equations form a special class of functional differential equations [3][5][6].

We assume that the right-hand side of (1.1) is continuous in all of its arguments and Lipschitz in z and $z(t-\tau)$. Then a solution is uniquely defined by specifying as initial condition a function segment z_0 whereby $z_0 \in \mathcal{C}([-\tau,\ 0],\ \mathbb{R}^n)$, the Banach space of continuous bounded functions mapping the delay-interval $[-\tau,\ 0]$ into $\mathbb{R}^n$ and equipped with the supremum-norm $\|.\|_s$.

Sufficient conditions for stability of a functional differential equation are provided by the theory of Lyapunov functionals [3] [6], a generalization of the classical Lyapunov theory for ODEs. For functional differential equations of the form

$$\dot{z} = F(z_t), \tag{2.1}$$

according to [3, Definition V.5.3], a mapping $V : \mathcal{C} \to \mathbb{R}$ is called a Lyapunov functional on a set G if V is continuous on $\overline{G}$ and $\dot{V} \leq 0$ on G. Here $\dot{V}$ is the upper-right-hand derivative of V along the solutions of (2.1), i.e.

$$\dot{V}(z_t) = \limsup_{h \to 0+} \frac{1}{h}\left[V(z_{t+h}) - V(z_t))\right].$$

The following theorem, taken from [3, Corollary V.3.1], provides sufficient conditions for stability:

THEOREM 2.1. *Suppose $z = 0$ is a solution of (2.1) and $V : \mathcal{C} \to \mathbb{R}$ is continuous with $V(0) = 0$. When there exist nonnegative functions $a(r)$ and $b(r)$ with $a(r) > 0$ as $r > 0$ and $a(r) \to \infty$ as $r \to \infty$ such that,*

$$a(\|z(t)\|) \leq V(z_t), \quad \dot{V}(z_t) \leq -b(\|z(t)\|).$$

Then the zero solution is stable and every solution is bounded. If in addition, $b(r)$ is positive definite, then every solution approaches zero as $t \to \infty$. Instead of working with functionals, it is also possible to use classical Lyapunov functions when relaxing the condition $\dot{V} \leq 0$. This approach, leading to the so-called Razumikhin-type theorems [6], is not considered in this paper.

In most of the theorems of the paper, the condition of global asymptotic stability for the unperturbed system (equation (1.1) with $\eta = 0$) is not sufficient. When the dimension of the system is higher than one, we sometimes need precise information about the interaction of different components of the state $z(t)$. This information is captured in the Lyapunov functional, associated with the unperturbed system. Therefore, when necessary, we will restrict ourself to a specific class of functionals, satisfying the following assumption:

ASSUMPTION 2.2. *The unperturbed system $\dot{z} = f(z, z(t-\tau))$ is* delay-independent *globally asymptotically stable (i.e. GAS for all values of the delay) with a Lyapunov-functional of the form*

$$V(z_t) = k(z) + \int_{t-\tau}^{t} l(z(\theta))d\theta \tag{2.2}$$

whereby $k(z) > 0$, $l(z) \geq 0$, $k(z)$ radially unbounded and such that the conditions of theorem 2.1 (with $b(r)$ positive definite) are satisfied. This particular choice is motivated by the fact that such functionals are used for a class of linear time-delay systems [3][6]. Furthermore choosing a delay-independent stable unperturbed system also allows us to investigate whether the results obtained in the presence of perturbations are still global in the delay. Note that in the ODE-case (2.2) reduces to $V = k(z)$ and hardly forms any restriction because under mild conditions its existence is guaranteed by converse theorems.

The perturbation $\eta(t) \in L_p([0,\ \infty))$ when $\exists M$ such that $\|\eta\|_p = \left[\int_0^\infty |\eta(s)|^p ds\right]^{\frac{1}{p}} = M < \infty$, $\eta(t) \in L_\infty$ when $\|\eta\|_\infty = \sup_{t\geq 0} |\eta(t)| < \infty$.

We assume η in (1.1) to be continuous and to belong to both L_1 and L_∞. When the perturbation is generated by an autonomous ODE, $\dot{\xi} = a(\xi)$, $\eta = b(\xi)$ with a and b continuous and locally Lipschitz, with $b(0) = 0$, which is globally asymptotically and locally exponentially stable (GAS and LES), these assumptions are satisfied.

In the paper we show that when the unperturbed system is delay-independent stable and the initial conditions bounded (i.e. $\|z_0\|_s \leq R < \infty$), arbitrarily small perturbations may cause unbounded trajectories provided the delay is large enough, hence arbitrarily small perturbations may destroy the delay-independent stability property. For such cases it is instructive to investigate whether semi-global stabilization in the delay is possible: with a parametrized perturbation $\eta(t,a)$, we say that the trajectories of (1.1) can be bounded semi-globally in z and semi-globally in the delay if for each compact region $\Omega \subset \mathbb{R}^n$, and $\forall \bar{\tau} \in \mathbb{R}^+$, there exists a positive number $\bar{a}$ such that all initial conditions $z_0 \in \mathcal{C}([-\tau,\ 0],\ \mathbb{R}^n)$, with $z_0(\theta) \in \Omega$, $\theta \in [-\tau,\ 0]$, $\forall \tau \leq \bar{\tau}$, give rise to bounded trajectories when $a \geq \bar{a}$.

A C^0 function $\gamma:\ \mathbb{R} \to \mathbb{R}$ belongs to class κ, if it is strictly increasing and $\gamma(0) = 0$. The symbol $\|.\|$ is used for the Euclidean norm in $\mathbb{R}^n$ and by $\|x,y\|$ we mean $\left(\|x\|^2 + \|y\|^2\right)^{\frac{1}{2}}$.

3. The mechanism of destabilizing perturbations. In contrast to linear systems, small perturbations (in the L_1 or L_∞ sense) are sufficient to destabilize nonlinear differential equations. In the ODE-case, the nonlinear mechanism for instability is well known: small perturbations suffice to make solutions escape to infinity in a finite time, for instance when the interconnection term Ψ is nonlinear in z. This is illustrated with the following example:

$$\begin{aligned} \dot{z} &= -z + z^2\eta \\ \dot{\eta} &= -a\eta, \end{aligned} \tag{3.1}$$

which can be solved analytically for z to give

$$z(t) = \frac{e^{-t}}{\frac{1}{z(0)} - \int_0^t e^{-s}\eta(s)ds} = \frac{e^{-t}}{\frac{1}{z(0)} - \eta(0)\int_0^t e^{-(1+a)s}ds}. \tag{3.2}$$

If $z(0)\eta(0) > 1 + a$, z escapes to infinity in a finite time t_e which is given by

$$t_e = \frac{1}{1+a}\log\left(\frac{z(0)\eta(0)}{z(0)\eta(0) - (1+a)}\right). \tag{3.3}$$

This last expression shows that the escape time becomes smaller as the initial conditions are chosen larger, and, as a consequence, however fast $\eta(t)$ would be driven to zero in the first equation of (3.1), $z(0)$ could always be chosen large enough for the solution to escape to infinity in finite time.

In the simple example (3.1), the perturbation is the output of a stable linear system. Its initial condition $\eta(0)$ dictates the L_∞-norm of the perturbation, while the parameter a controls its L_1-norm. Making these norms arbitrarily small does not result in global stability. This is due to the nonlinear growth of the interconnection term.

One may wonder whether the instability mechanism encountered in the ODE situation (3.1) will persist in the DDE situation

$$\left\{ \begin{array}{l} \dot{z} = -bz + z(t-\tau)^2\eta \\ \dot{\eta} = -a\eta \end{array} \right. . \tag{3.4}$$

In contrast to (3.1), system (3.4) exhibits no finite escape time. This can be proven by application of the method of steps, i.e. from the boundedness of $z(\theta)$, $\theta \in [(k-1)\tau, k\tau]$, we conclude boundedness in $[k\tau, (k+1)\tau]$ of $\dot{z}(\theta)$ and thus of $z(\theta)$. Nevertheless the exponentially decaying input η still causes unbounded solutions in (3.4): this particular system is easily seen to have an exponential solution $z_e(t) = \frac{a+b}{\eta(0)} e^{2a\tau} e^{at}$. The instability mechanism can be explained by the superlinear divergence of the solutions of $\dot{z} = z^\alpha(t-\tau)$ for $\alpha > 1$:

PROPOSITION 3.1.

$$\dot{z} = z(t-\tau)^\alpha, \quad \alpha > 1$$

has solutions which diverge faster than any exponential function.

Proof. Take as initial condition a strictly positive solution segment z_0 over $[-\tau,\ 0]$ with $z(0) > 1$. For $t \geq 0$, the trajectory is monotonically increasing. This means that in the interval $[k\tau,\ (k+1)\tau]$ for $k \geq 1$,

$$z((k-1)\tau)^\alpha \leq \dot{z} \leq z(k\tau)^\alpha.$$

The solution at point $k\tau, k \geq 1$ is bounded below by the sequence satisfying

$$z_{k+1} = z_k + \tau z_{k-1}^\alpha, \quad z_0 = z(0),\ z_1 = z(\tau).$$

which has limit $+\infty$. The ratio $R_k = \frac{z_k}{z_{k-1}}$ satisfies

$$R_{k+1}R_k = R_k + \tau z_{k-1}^{\alpha-1}.$$

and consequently $(R_{k+1} - 1)R_k$ tends to infinity. However for an exponential function e^{at}, $R = e^{a\tau}$ and $(R-1)R$ is constant. ☐

Because of the faster than exponential growth of z in (3.4), arbitrarily fast exponential decay of η cannot counter the blow-up caused by the nonlinearity in $z(t-\tau)$, and hence the system is not globally asymptotically stable.

The instability mechanism illustrated by (3.1) and (3.4) can be avoided by imposing suitable growth restrictions on the interconnection term Ψ. When the unperturbed system is scalar, it is sufficient to restrict the interconnection term to have linear growth in both of its arguments, i.e.

$$\exists c_1, c_2 > 0 \text{ such that } \|\Psi(z, z(t-\tau))\| \leq c_1 + c_2\|z, z(t-\tau)\|. \tag{3.5}$$

This linear growth condition is by itself not sufficient however, if the unperturbed system has dimension greater than one. In that case, the interaction of the different components of the state $z(t)$ can still cause "nonlinear" effects leading to unbounded solutions. An illustration of this phenomenon is given by the following system

$$\left\{ \begin{array}{lcl} \dot{z}_1 & = & -z_1 + z_2\eta(t) \\ \dot{z}_2 & = & -z_2 + z_1^2 z_2 \\ \dot{\eta} & = & -\eta. \end{array} \right. \tag{3.6}$$

which was shown in [13] to have unbounded solutions, despite the linearity of the interconnection. The instability is caused by the mutual interaction between z_1 and z_2 when $\eta \neq 0$.

The following theorem, inspired by Theorem 4.7 in [13], provides sufficient conditions for bounded solutions. To prevent the instability mechanism due to interacting states, conditions are put on the Lyapunov functional of the unperturbed system.

THEOREM 3.2. *Assume that the system $\dot{z} = f(z, z(t-\tau)) + \Psi(z, z(t-\tau))\eta$ satisfies Assumption 2.2 and that the interconnection term $\Psi(z, z(t-\tau))$ grows linearly in its arguments, i.e. satisfies (3.5). Furthermore if the perturbation $\eta(t) \in L_1([0,\infty))$ and $k(z)$ satisfies:*

(i) $\alpha_1\|z\|^\gamma \leq k(z) \leq \alpha_2\|z\|^\gamma, \quad 0 < \alpha_1 < \alpha_2 < \infty, \quad 1 \leq \gamma < \infty,$

(ii) $\|\frac{dk}{dz}\| \, \|z\| \leq ck(z),$

then all trajectories of the perturbed system are bounded, for all values of the time delay. Condition (ii) is sometimes called a *polynomial growth condition* because it is satisfied if $k(z)$ is polynomial in z, but not satisfied if $k(z)$ is exponential in z.

Proof. Along a trajectory $z(t)$ we have:

$$\begin{aligned} \dot{V} &\leq \|\tfrac{dk}{dz}\| \|\Psi(z, z(t-\tau))\| |\eta| \\ &\leq c\tfrac{k(z)}{\|z\|}\left(c_1 + c_2\sqrt{\|z\|^2 + \|z(t-\tau)\|^2}\right)|\eta| \\ &\leq c\alpha_2^{1/\gamma}k(z)^{1-1/\gamma}\left(c_1 + c_2\sqrt{\tfrac{k(z)^{2/\gamma}}{\alpha_1^{2/\gamma}} + \tfrac{k(z(t-\tau))^{2/\gamma}}{\alpha_1^{2/\gamma}}}\right)|\eta| \\ &\leq c\alpha_2^{1/\gamma}\left(c_1 k(z)^{1-1/\gamma} + c_2\alpha_1^{-1/\gamma}\sqrt{k(z)^2 + k(z)^{2-2/\gamma}k(z(t-\tau))^{2/\gamma}}\right)|\eta| \\ &\leq c\alpha_2^{1/\gamma}\left(c_1 V^{1-1/\gamma} + c_2\alpha_1^{-1/\gamma}\sqrt{V^2 + V^{2-2/\gamma}k(z(t-\tau))^{2/\gamma}}\right)|\eta|. \end{aligned} \tag{3.7}$$

For $t \in [0,\ \tau]$, $z(t)$ cannot escape to infinity because $k(z(t-\tau))$ is bounded (calculated from the initial condition) and the above estimate can be integrated over the interval since the right hand side is linear in V and $\eta \in L_1$.

For $t \geq \tau$ we can use the estimate $k(z(t-\tau)) \leq V(z(t-\tau))$:

$$\dot{V} \leq c\alpha_2^{1/\gamma}\left(c_1 V^{1-1/\gamma} + c_2\alpha_1^{-1/\gamma}\sqrt{V^2 + V^{2-2/\gamma}V(t-\tau)^{2/\gamma}}\right)|\eta|.$$

Because this estimate for $\dot{V}$ is increasing in both of its argument, an upper bound for $V(t)$ along the trajectory is described by

$$\dot{W} = c\alpha_2^{1/\gamma}\left(c_1 W^{1-1/\gamma} + c_2\alpha_1^{-1/\gamma}\sqrt{W^2 + W^{2-2/\gamma}W(t-\tau)^{2/\gamma}}\right)|\eta|$$

with as initial condition $W(z_\tau) = V(z_\tau)$. Via the method of steps, it is clear that W cannot escape to infinity in a finite time. From $t = \tau$ on, W is monotonically increasing. As a consequence, for $t \geq 2\tau$, $W(t) \geq W(t-\tau)$ and

$$\dot{W} \leq c\alpha_2^{1/\gamma}\left(c_1 W^{1-1/\gamma} + c_2\alpha_1^{-1/\gamma}\sqrt{2}W\right)|\eta(t)|,$$

and this estimate can be integrated leading to boundedness of $\lim_{t\to\infty} \sup V(t)$ because $\eta(t) \in L_1$. Hence the trajectory $z(t)$ is bounded. □

REMARK 3.3. *When the interconnection term is undelayed, i.e. Ψ only depends on the argument z, condition (i) in Theorem 3.2 can be dropped, and as a special case (also f undelayed), Theorem 4.7 of [13] is recovered. The presence of a delay in the unperturbed system does not provide extra complications compared to the ODE-case and the proof is analogous to the proof of Theorem 4.7 of [13]:*

Along a trajectory, we now have,

$$\dot{V} \leq \|\frac{dk}{dz}\|(c_1 + c_2\|z\|)|\eta|$$

When $\|z\| \geq 1$ it follows from $\|\frac{dk}{dz}\| \leq \frac{ck(z)}{\|z\|}$ that $\dot{V} \leq c(c_1+c_2)V|\eta|$. When $\|z\| \leq 1$, we have $\dot{V} \leq M|\eta|$ with $M = \sup_{\|z\|\leq 1} \|\frac{dk}{dz}\|(c_1 + c_2\|z\|)$, and when in addition $V \geq 1$, we have $\dot{V} \leq MV|\eta|$.

Hence the following estimate holds whenever $V \geq 1$,

$$\dot{V} \leq \max\left(c(c_1+c_2), M\right) V|\eta|.$$

From the explicit integration of this estimate the boundedness of V and the trajectory is proven. □

4. Semi-global results for controlled perturbations. Although no global results can be guaranteed in the absence of growth conditions, the examples in the previous section suggest that one should be able to bound the solutions semi-globally in the space of initial conditions by decreasing the size of the perturbation. Therefore we assume that the perturbation is parametrized,

$$\eta = \eta(t,a).$$

We will consider two cases: a) parameter a controls the L_1- or the L_∞-norm of η and b) a regulates the shape of a perturbation with fixed L_1-norm.

4.1. Controlling the L_1 and the L_∞ norm of the perturbation. We first assume that the L_1-norm of η is controlled. We have the following result:

THEOREM 4.1. *Consider the system*

$$\dot{z} = f(z, z(t-\tau)) + \Psi(z, z(t-\tau))\eta(t,a),$$

and suppose that the unperturbed system is GAS with the Lyapunov functional $V(z_t)$ satisfying Assumption 2.2. If furthermore $\|\eta(t,a)\|_1 \to 0$ as $a \to \infty$, then the trajectories can be bounded semi-globally both in z and the delay τ, by increasing a.

Proof. Let $\tau \geq 0$ be fixed and denote by Ω the desired stability domain in $\mathbb{R}^n$, i.e. such that all trajectories starting in z_0 with $z_0(\theta) \in \Omega$ for $\theta \in [-\tau,\ 0]$ are bounded. Let $V_c = \sup_{z_0\in\Omega} V(z_0)$. We have

$$\begin{aligned}\dot{V}(z_t) &\leq k'(z)f(z,z(t-\tau)) + l(z(t)) - l(z(t-\tau)) + k'(z)\Psi(z,z(t-\tau))\eta(t,a)\\ &\leq |k'(z)\Psi(z,z(t-\tau))|.|\eta(t,a)|.\end{aligned}$$

As long as $V(t) \leq 2V_c$, $z(t)$ and $z(t-\tau)$ belong to a compact set. Hence $\exists M > 0$ such that $|k'(z)\Psi(z,z(t-\tau)| \leq M$ and

$$V(t) - V(0) \leq M\int_0^\infty |\eta(s,a)|ds = M\|\eta(t,a)\|_1.$$

When $a \to \infty$, the increase of V tends to zero. As a consequence the assumption $V(t) \leq 2V_c$ is valid for $\forall t \geq 0$. Hence the trajectories with initial condition in Ω are bounded.

Note that for a fixed region $\Omega \in \mathbb{R}^n$, V_c increases with τ and this influences both the value M in the estimation of $|k'(z)\Psi(z, z(t-\tau)|$ and the critical value $\bar{a}$ of a in order to bound the trajectories. However when τ belongs to a compact interval $[0,\ \bar{\tau}]$, we can take $a \geq \sup_{\tau \in [0,\ \bar{\tau}]} \bar{a}(\tau)$ and hence bound the trajectories semi-globally in both the state and the delay. $\square$

The result given above is natural because for a given initial condition, a certain amount of energy is needed for destabilization, expressed mathematically by $\|\eta\|_1$. However global stability in the state is not possible because the required energy can become arbitrarily small provided the initial condition is large enough, see for instance example (3.1). Later we will discuss why the trajectories cannot be bounded globally in the delay.

Now we consider the case whereby the L_∞-norm of the perturbation is controlled.

THEOREM 4.2. *Consider the system*

$$\dot{z} = f(z, z(t-\tau)) + \Psi(z, z(t-\tau))\eta(t, a)$$

Suppose that the unperturbed system is GAS with the Lyapunov functional $V(z_t)$ satisfying Assumption 2.2. If $\|\eta(t,a)\|_\infty \to 0$ as $a \to \infty$, the trajectories of the perturbed system can be bounded semi-globally in both z and the delay τ.

Proof. As in the proof of Theorem 4.1, it is sufficient to prove semi-global stability in the state for a fixed $\tau \geq 0$. Let Ω and V_c be defined as in Theorem 4.1. Define $\Omega_2 = \{z \in \mathbb{R}^n : k(z) \leq 4W_c\}$ and $\Omega_\epsilon = \{z : \|z\| \leq \epsilon\} \subset \Omega$, with $\epsilon > 0$ small.

The time derivative of V satisfies

$$\begin{aligned} \dot{V}(z_t) &= k'(z)f(z, z(t-\tau)) + l(z(t)) - l(z(t-\tau)) + k'(z)\Psi(z, z(t-\tau))\eta(t, a) \\ &\leq -b(\|z\|) + |k'(z)\Psi(z, z(t-\tau))|\eta(t, a). \end{aligned} \tag{4.1}$$

Let $M = \sup_{z,y \in \Omega_2} |k'(z)\Psi(z, y)|$.

When $z(t) \in \Omega_2 \backslash \Omega_\epsilon$ we have, since b is positive definite, $\dot{V} \leq -b(\|z\|) + M\|\eta\|_\infty \leq -N$ for some number $N > 0$ provided $\|\eta\|_\infty$ is small enough. Only when $z(t) \in \Omega_\epsilon$, the value of V can increase with the estimate $\dot{V} \leq M\|\eta\|_\infty$.

Now we prove by contradiction that all trajectories with initial condition in Ω are bounded for small $\|\eta\|_\infty$: suppose that a solution starting in Ω (with $V \leq V_c$) is unbounded. Then it has to cross the level set $2V_c$. Assume that this happens for the first time at t^*. Note that for small $\|\eta\|_\infty$, t^* is large. During the interval $[t^* - \tau,\ t^*]$, V can both increase and decrease, but $V(t^*) > V(t^* - \tau)$. While V increases, $z(t) \in \Omega_\epsilon$ and the increase ΔV is limited: $\Delta V \leq M\|\eta\|_\infty \tau$. When $z(t)$ would be outside Ω_ϵ for a time-interval $\Delta t \subset [t^* - \tau,\ t^*]$, whereby $\dot{V} \leq -N$, we have:

$$N\Delta t \leq M\tau\|\eta\|_\infty. \tag{4.2}$$

Hence by reducing $\|\eta(t,a)\|_\infty$ we can make the time-interval Δt arbitrarily small. On the other hand (for large a),

$$\|\frac{dz}{dt}\| \leq \|f(z, z(t-\tau)) + \Psi(z, z(t-\tau))\eta(t, a)\| \leq L < \infty$$

when z_t is inside Ω_2, because f and Ψ map bounded sets into bounded sets. Hence with $|t_2 - t_1| \leq \Delta t$ we have $\|z(t_1) - z(t_2)\| \leq L\Delta t$. Because of (4.2) we can increase

a (reduce $\|\eta(t,a)\|_\infty$) such that $L\Delta t \leq \epsilon$ and consequently we have:

$$\|z(t)\| \leq 2\epsilon, \quad t \in [t^* - \tau,\ t^*].$$

If ϵ was chosen such that $\Omega_{2\epsilon}$ lies inside Ω, we have a contradiction because this implies $W(t^*) \leq W_c$. Hence a trajectory can never cross the level set $2W_c$ and is bounded. □

The results of Theorems 4.1 and 4.2 are not global in the delay, even though the unperturbed system is delay-independent stable. Global results in the delay are generally not possible: we now give an example where it is impossible to bound the trajectories semi-globally in the state and globally in the delay, even if we make the size of the perturbation arbitrarily small w.r.t. the L_1 and L_∞-norm.

EXAMPLE 4.3. *Consider the following system:*

$$\left\{ \begin{array}{l} \dot{z_1} = -2z_1 + z_1(t-\tau) \\ \dot{z_2} = -\frac{(z_1-2)^2-1}{z_2^2+1} z_2 + z_2^3\eta(t,a) \end{array} \right. . \tag{4.3}$$

The unperturbed system, i.e. (4.3) with $\eta = 0$, is delay-independent stable. This is proven with the Lyapunov functional

$$V = z_1^2 + \int_{t-\tau}^{t} z_1^2 d\theta + \frac{1}{2} z_2^2.$$

Its time derivative

$$\begin{aligned} \dot{V} \;=\; & [z_1\ z_1(t-\tau)] \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_1(t-\tau) \end{bmatrix} \\ & -z_2^2 \tfrac{(z_1-2)^2-1}{z_2^2+1} \\ & \leq -2z_1^2 - z_2^2 \tfrac{(z_1-2)^2-1}{z_2^2+1}. \end{aligned}$$

is negative definite: when $z_1 \notin [1,\ 3]$, both terms are negative and in the other case the second term is dominated, because it saturates in z_2. From this it follows that the conditions of Assumption 2.2 are satisfied.

With the perturbation

$$\begin{array}{rll} \eta(t,a) & = (t-t_0)^2 e^{-a(t-t_0)} & t \geq t_0 = a^3 \\ & = 0 & t \leq t_0 \end{array} , \tag{4.4}$$

whereby increasing a leads to a reduction of both $\|\eta\|_1$ and $\|\eta\|_\infty$, we can not bound the trajectories semi-globally in the state and globally in τ: for each value of a we can find a bounded initial condition (upper bound independent of a), leading to a diverging solution, provided τ is large enough: the first equation of (4.3) has a solution $z_1(t) = 2.5e^{-\alpha t}$ whereby $-\alpha$ is the real solution of equation

$$\lambda = -2 + e^{-\lambda\tau}.$$

Clearly $\alpha \to 0$ as $\tau \to \infty$. Since $z_1(-\tau) = 2.5e^{\alpha\tau} \to 5$ as $\tau \to \infty$, uniform boundedness in τ of this solution over the interval $[-\tau,\ 0]$ (initial condition) is guaranteed. Choose $z_2(0) = 1$.

The above solution for z_1 satisfies:

$$z_1(t) \in [1.5,\ 2.5]$$

when $t \in [0,\ \frac{1}{\alpha}\log\frac{5}{3}]$ *and thus*

$$\dot{z}_2 \geq \frac{z_2}{2(1+z_2^2)} + z_2^3\eta(t,a). \tag{4.5}$$

A rather lengthy calculation shows that with $z_2(0) = 1$ *and the perturbation (4.4), the solution of (4.5) always escapes to infinity in a finite time* $t_f(a)$. *Hence this also holds for the solution of the original system when the delay is large enough such that*

$$\frac{1}{\alpha(\tau)}\log\frac{5}{3} > t_f(a).$$

This result is not in contradiction with the intuition that a perturbation with small L_1-norm can only cause escape in a finite time when the initial condition is far away from the origin, as illustrated with example (3.1): in the system (4.3) with $\eta = 0$, z_2 is driven away from the origin as long as $z_1 \in [1,\ 3]$. By increasing the delay in the first equation, we can keep z_1 in this interval as long as desired. Thus the diverging transient of the *unperturbed* system is used to drive the state away from the origin, far enough to make the perturbation cause escape.

4.2. Controlling the shape of the perturbation. We assume that the shape of a perturbation with a fixed L_1-norm can be controlled and consider the influence of a concentration of the perturbation in arbitrarily small time-intervals near $t = 0$. In the ODE case this does not allow to improve stability properties. This is illustrated with the first equation of example (3.1): instability occurs when $z(0) \geq \frac{1}{\int_0^t e^{-s}\eta(s)ds}$ and by concentrating the perburbation the stability domain may even shrink, because the beneficial influence of damping is reduced. In the DDE-case however, when the interconnection term is linear in the undelayed argument, it behaves as linear during one delay interval preventing escape. Moreover, starting from a compact region of initial conditions, the reachable set after one delay interval can be bounded independently of the shape of the perturbation (because of the fixed L_1-norm). After one delay interval we are in the situation treated in Theorem 4.1. This is expressed in the following theorem. As in Theorem 3.2 the polynomial growth condition prevents a destabilizing interaction between different components of the state vector $z(t)$.

THEOREM 4.4. *Consider*

$$\dot{z}(t) \;=\; f(z(t), z(t-\tau)) + \Psi(z(t), z(t-\tau))\eta(t,a) \tag{4.6}$$

and suppose that the unperturbed system is GAS with the Lyapunov functional $V(z_t) = k(z) + \int_{t-\tau}^{t} l(z(\theta))d\theta$ *satisfying Assumption 2.2. Let* $k(z)$ *satisfy the polynomial growth condition* $\|\frac{dk}{dz}\|\|z\| \leq ck(z)$. *Assume that* Ψ *has linear growth in* $z(t)$, $\|\eta(t,a)\|_1 < \infty$ *is independent of* a *and* $\lim_{a\to\infty}\int_t^\infty |\eta(s,a)|ds = 0, \forall t > 0$. *Then the trajectories of (4.6) can be bounded semi-globally in* z *and for all* $\tau \in [\tau_1,\ \tau_2]$ *with* $0 < \tau_1 \leq \tau_2 < \infty$.

Proof. Consider a fixed $\tau \in [\tau_1,\ \tau_2]$ and let Ω be the desired stability domain in $\mathbb{R}^n$ and let R be such that $z_0(\theta) \in \Omega,\ \forall\theta \in [-\tau,\ 0] \Rightarrow ||z_0||_s \leq R$.

The interconnection term has linear growth in z, i.e. there exist two class-κ functions γ_1 and γ_2 such that

$$\|\Psi(z, z(t-\tau))\| \leq \gamma_1(\|z(t-\tau)\|) + \gamma_2(\|z(t-\tau)\|)\|z\|.$$

The time-derivative of the Lyapunov function V satisfies

$$\begin{array}{ll} \dot{V} & \leq \|\frac{dk}{dz}\|.\|\Psi(z, z(t-\tau))\|.|\eta(t,a)| \\ & \leq \|\frac{dk}{dz}\|.\left(\gamma_1(\|z(t-\tau)\|) + \gamma_2(\|z(t-\tau)\|)\|z\|\right).|\eta(t,a)| \\ & \leq ck(z).\left(\frac{\gamma_1(\|z(t-\tau)\|)}{\|z\|} + \gamma_2(\|z(t-\tau)\|)\right)|\eta(t,a)| \\ & \leq cV(z_t)|\eta(t,a)|\left(\frac{\gamma_1(\|z(t-\tau)\|)}{\|z\|} + \gamma_2(\|z(t-\tau)\|)\right) \end{array}$$

During the interval $[0,\ \tau]$, $z(t-\tau)$ belongs to Ω. Therefore, when $\|z\| \geq R$, one can bound $\left(\frac{\gamma_1(\|z(t-\tau)\|)}{\|z\|} + \gamma_2(\|z(t-\tau)\|)\right)$ by a factor c_2 independent of a. Thus

$$\dot{V} \leq cc_2 V|\eta(t,a)|,$$

and when a trajectory leaves the set $\{z : \|z\| \leq R\}$ at t^*, because $\eta \in L_1$,

$$\begin{array}{ll} V(t) & \leq V_{\max} e^{cc_2 \int_{t^*}^{t} |\eta(s,a)|ds} \\ & \leq V_{\max} e^{cc_2\|\eta\|_1} = M \end{array}$$

for some constant M, independently of a. In the above expression, $V_{\max} = \sup_{\|z_t\|_s \leq R} V(z_t)$.

As a consequence, also $k(z)$ and $\|z(t)\|$ can be bounded, uniformly in $t \in [0,\ \tau]$ and a. Hence at time τ the state z_τ, i.e. $z(t),\ t \in [0, \tau]$ belongs to a compact region Ω_2 independently of a.

Now we can translate the original problem over one delay interval: at time τ the initial conditions belong to the bounded region Ω_2 and with $t' = t - \tau$ we have:

$$\|\eta(t', a)\|_1 = \int_{\tau}^{\infty} |\eta(s,a)|ds \leq \int_{\tau_1}^{\infty} |\eta(s,a)|ds \to 0 \text{ as } a \to \infty.$$

Because of Theorem 4.1, we can increase a such that all solutions starting in Ω_2 are bounded.

Until now we assumed a fixed τ. But because $[\tau_1,\ \tau_2]$ is compact, we can take the largest threshold of a for bounded solutions over this interval. □

REMARK 4.5. *Whenever the perturbation in (1.1) is generated by a GAS ODE, the boundedness results are strengthened to asymptotic stability results. This can be shown following the lines of the proof of Proposition* 4.1 *in [13]. Stability follows from a local version of Theorem 4.1 and attractivity from the application of a generalization to the time-delay case of the classical LaSalle's Theorem [3, Theorem V.*3.1*].*

5. Stabilization of partially linear cascades. In the rest of the paper we consider the stabilization of the cascade (1.2) with the control law (1.3)

From the previous sections it is clear that the input y of the z-subsystem can act as a destabilizing disturbance. However, the control can drive the output of the linear system fast to zero. We will investigate under which conditions this is sufficient to stabilize the whole cascade. An important issue in this context is the so-called *fast peaking* phenomenon [15]. This is a structural property of the ξ-system whereby imposing faster convergence of the output to zero implies larger overshoots which can in turn destabilize the cascade and may form an obstacle to both global and semi-global stabilizability. We start with a short description of the peaking phenomenon and then apply the results of the previous section to the stabilization of the cascade system.

Our presentation of the peaking phenomenon is inspired by [15] but, following [13], we place the phenomenon in an input-output framework rather than an input state framework. We also emphasize the relation with between a peaking system and the L_1-norm of its output.

5.1. The peaking phenomenon. When in the system,

$$\begin{aligned} \dot{\xi} &= A\xi + Bu \\ y &= C\xi, \end{aligned} \tag{5.1}$$

the pair (A, B) is controllable, one can always find state feedback laws $u = F\xi$ resulting in an exponential decay rate with exponent $-a$. Then the output of the closed loop system satisfies

$$\|y(t)\| \leq \gamma\|\xi(0)\|e^{-at}, \tag{5.2}$$

where γ depends on the choice of the feedback gain. We are interested in the lowest achievable value of γ among different feedback laws and its dependence upon a. This will be determined by the so-called peaking exponent, which we now define.

Denote by $\mathcal{F}(a)$ the collection of all stabilizing feedback laws $u : \xi \to F\xi$ with the additional property that all *observable** eigenvalues λ of (C, A_F), with $A_F = A + BF$, satisfy $\mathrm{Re}(\lambda) < -a$. For a given a and $F \in \mathcal{F}(a)$, define the smallest value of γ in (5.2) as

$$\kappa_F(a) = \sup\left\{\|y(t)\|e^{at}\right\},$$

where the supremum is taken over all $t \geq 0$ and all initial conditions satisfying $\|\xi(0)\| \leq 1$. Now denote by $\kappa(a) = \inf_{F\in\mathcal{F}(a)} \kappa_F$. The output of system (5.1) is said to have peaking exponent s when there exists constants α_1, α_2 such that

$$\alpha_1 a^s < \kappa(a) < \alpha_2 a^s \tag{5.3}$$

for large a. When $s = 0$ the output is said to be *nonpeaking.*

The peaking exponent s is a structural property related to the zero-dynamics: when the system has relative degree r, it can be transformed (including a preliminary feedback transformation) into the normal form [4][1]:

$$\left\{ \begin{array}{l} \dot{\xi}_0 = A_0\xi_0 + B_0 y \\ y^{(r)} = u, \end{array} \right. \tag{5.4}$$

which can be interpreted as an integrator chain linearly coupled with the zero-dynamics subsystem $\dot{\xi}_0 = A_0\xi_0$. Using state feedback the output of an integrator chain can be forced to zero rapidly without peaking [13]. Because of the linear interconnection term, stability of the zero-dynamics subsystem implies stability of the whole cascade. On the contrary, when the zero dynamics are unstable, some amount of energy, expressed by $\int_0^\infty \|y(t)\|dt$, is needed for its stabilization and therefore the output must peak. More precisely we have the following theorem, proven in the appendix.

THEOREM 5.1. *The peaking exponent s equals the number of eigenvalues in the closed RHP of the zero-dynamics subsystem.* The definition of the peaking exponent (5.3) is based on an upper bound of the exponentially weighted output, while its L_1-norm is important in most of the theorems of Section 4. But because the overshoots related to peaking occur in a fast time-scale ($\sim at$), there is a connection. For instance we have the following theorem, based on a result of Braslavsky and Middleton [10]:

THEOREM 5.2. *When the output y of system (5.1) is peaking ($s \geq 1$), $\|y(t)\|_1$ can not be reduced arbitrarily.*

*In [15], where the peaking phenomenon is rather studied in an input-state framework, one places all eigenvalues to the left of the line $\lambda = -a$.

Proof. Denote by z_0 an unstable eigenvalue of the zero-dynamics of (5.1). When a feedback $u = F\xi$ is stabilizing the relation between y and $w = u + F\xi$ in the Laplace-domain is given by

$$\begin{aligned} Y(s) &= C(sI - \bar{A})^{-1}BW(s) + C(sI - \bar{A})^{-1}\xi(0) \\ &= H(s)W(s) + C(sI - \bar{A})^{-1}\xi(0), \end{aligned}$$

with $\bar{A} = A + BF$. The first term vanishes at z_0 because the eigenvalues of the zero dynamics appear as zeros in the corresponding transfer function $H(s)$ and since the feedback F is stabilizing, no unstable pole-zero cancellation occurs at z_0. Hence

$$\begin{aligned} \|y\|_1 &\geq \int_0^\infty |y(t)e^{-z_0 t}|dt \\ &\geq |\int_0^\infty y(t)e^{-z_0 t}dt| \\ &= |C(z_0 I - \bar{A})^{-1}\xi(0)| \end{aligned} \quad . \tag{5.5}$$

□

5.2. Nonpeaking cascades. When the ξ-subsystem is minimum-phase and thus nonpeaking, one can find state feedback laws $u = F_a\xi$ resulting in

$$|y(t)| \leq \alpha_2 e^{-at}$$

and the L_1-norm of the output can be made arbitrarily small. So by Theorem 4.1, the cascade (1.2) can be stabilized semi-globally in the state and in the delay.

5.3. Peaking cascades. When the ξ-subsystem is nonminimum phase, the peaking phenomenon forms an obstacle to semi-global stabilizability, because the L_1-norm of the output cannot be reduced (Theorem 5.2).

For ODE-cascades, we illustrate the peaking obstruction with the following example:

EXAMPLE 5.3. *In the cascade,*

$$\begin{aligned} &\dot{z} = -z + z^2 y \\ &\dot{\xi}_1 = \xi_1 + \xi_2 \\ &\dot{\xi}_2 = u, \quad y = -\xi_2 \end{aligned} \quad ,$$

the peaking exponent of the ξ-subsystem is 1 (zero dynamics $\dot{\xi}_1 = \xi_1$). The cascade cannot be stabilized semi-globally since the explicit solution of the first equation is given by

$$z(t) = \frac{e^{-t}}{\frac{1}{z(0)} - \int_0^t e^{-s}y(s)ds}$$

whereby $\int_0^\infty e^{-s}y(s)ds = \xi_1(0)$. Hence the solution reaches infinity in a finite time when $0 < \frac{1}{\xi_1(0)} < z(0)$.

For DDE-cascades, we consider two cases:

Case 1: Peaking exponent=1. We can apply theorem 4.4 and obtain semi-global stabilizability in the state and in the delay, when the interconnection term is linear in the undelayed argument: besides (5.5) the L_1-norm of y can also be bounded from above since there exists feedback laws $u = F_a\xi$ such that

$$\|y(t)\|_1 \leq \int_0^\infty \alpha_2 a e^{-at} = \alpha_2,$$

and because of the fast time-scale property, the energy can be concentrated since $\forall t > 0$:

$$\int_t^\infty |y(s)|ds \leq \int_t^\infty \alpha_2 a e^{-as} ds \to 0 \text{ as } a \to \infty.$$

Case 2: Peaking exponent > 1. In this case, we expect the L_1-norm of y to grow unbounded with a, as suggested by the following example:

EXAMPLE 5.4. *When ξ_k is considered as the output of the integrator chain,*

$$\dot{\xi}_1 = \xi_2,\ \dot{\xi}_2 = \xi_3, \ldots, \dot{\xi}_n = u,$$

the peaking exponent is $k-1$ (Theorem 5.1) and $\|x_k(t)\|_1,\ k = 2, n$ cannot be reduced arbitrarily by achieving a faster exponential decay rate. In Proposition 4.32 of [13], it is shown that with the feedback-law $u = K(a)\xi = -\sum_{k=1}^n a^{n-k+1} q_{k-1}\xi_k$, where all solutions of $\sum_{k=0}^{n-1} q_k \lambda^k + \lambda^n = 0$ satisfy $\mathrm{Re}(\lambda) < -1$, there exists a constant c independent of a such that

$$|\xi_k(t)| \leq c a^{k-1} e^{-at} \|\xi(0)\|,$$

hence the particular feedback $u = K(a)\xi$ is able to achieve an upper bound which corresponds to definition (5.3), for each choice of the output $y = \xi_k$. It is also shown in [13] that with the same feedback and with as initial condition $\xi_1(0) = 1,\ \xi_k(0) = 0,\ k = 2, n$, there exists a constant d such that $s_k = \sup_{t \geq 0} |\xi_k(t)| \geq d a^{k-1}$. Define $t_k,\ k = 2, n$ such that $|\xi_k(t_k)| = s_k$. As a consequence,

$$\|\xi_k(t)\|_1 \geq \left| \int_0^{t_{k-1}} \xi_k(s) ds \right| = |\xi_{k-1}(t_{k-1})| \geq d a^{k-2}, \quad k = 3, n$$

while the peaking exponent of output $y = \xi_k$ is $k-1$.

With the two following examples, we show that when the energy of an exponentially decaying input perturbation ($\sim e^{-at}$) grows unbounded with a, an interconnection term which is linear in the undelayed argument, is not sufficient to bound the solutions semi-globally in the state. Because it is hard to deal in general with outputs generated by a linear system with peaking exponent $s > 1$, we use an artificial perturbation $a^s e^{-at}$, which has both the fast time-scale property and the suitable growth-rate of the energy (a^{s-1}) w.r.t. a.

EXAMPLE 5.5. *The solutions of equation*

$$\dot{z} = -bz + z z(t-\tau)^\alpha a^s e^{-at}, \quad \alpha > 0 \tag{5.6}$$

can not be bounded semi-globally in z by increasing a, for any $\tau > 0$, if the 'peaking exponent' s is larger than one.

Proof. Equation (5.6) has an exponential solution $z_e(t)$,

$$z_e(t) = \left[\frac{(\frac{a}{\alpha} + b)e^{a\tau}}{a^s} \right]^{\frac{1}{\alpha}} e^{\frac{a}{\alpha}t}.$$

Consider the solution $z(t)$ with initial condition $z_0 \equiv L > 0$ on $[-\tau,\ 0]$. For $t \in [0,\ \tau]$, $z(t)$ satisfies:

$$\dot{z} = -bz + z L^\alpha a^s e^{-at}$$

and consequently coincides on $[o,\ \tau]$ with

$$y(t) = Le^{L^\alpha a^{s-1}(1-e^{-at})-bt} \tag{5.7}$$

For large a, expression (5.7) describes a decreasing lower bound on $[\tau,\ 2\tau]$, since $y(t)$ reaches its maximum in $t^*(a)$ with $t^* \to 0$ as $a \to \infty$. Thus imposing $y(2\tau) > z_e(2\tau)$ implies that $z(t) > z_e(t), t \in [\tau,\ 2\tau]$ and from this one can argue [†] that $z(t) \geq z_e(t),\ t \geq \tau$. Thus the trajectory starting with initial condition L on $[-\tau,\ 0]$ is unbounded when

$$Le^{L^\alpha a^{s-1}(1-e^{-2a\tau})-2b\tau} > x_e(2\tau) = \left[\frac{\frac{a}{\alpha}+b}{a^s}\right]^{\frac{1}{\alpha}} e^{\frac{3a}{\alpha}\tau}.$$

When $s > 2$, for each value of L, the solution is unstable for large a, thus the attraction domain of the stable zero solution shrinks to zero. When $s = 2$, a solution starting from $L > \left[\frac{3\tau}{\alpha}\right]^{\frac{1}{\alpha}}$ is unstable for large a. □

Even when the interconnection term contains no terms in $z(t)$, but only delayed terms of z, semi-global results are still not possible in general, as shown with the following example.

EXAMPLE 5.6. *The solutions of the system*

$$\dot{z} = -\mathrm{sat}(z) + e^{z(t-\tau)}a^s e^{-at}, \tag{5.8}$$

with $\mathrm{sat}(z) = z$ *when* $|z| \leq 1$ *and* $\mathrm{sat}(z) = \mathrm{sign}(z)$ *otherwise, can not be bounded semi-globally in* z *by increasing* a*, for any* $\tau > 0$*, when the 'peaking exponent'* s *is greater than one.*

Proof. When $z \geq 1$, equation (5.8) reduces to:

$$\dot{z} = -1 + e^{z(t-\tau)}a^s e^{-at},$$

which has the following explicit solution,

$$z_l(t) = at + b, \quad b = a\tau - \log(\frac{a^s}{a+1}).$$

When the initial condition of (5.8) is L on $[-\tau,\ 0]$, during one delay-interval, one can find an lower bound of the solution by integrating

$$\dot{z} = -z + e^L a^s e^{-at}$$

with solution

$$z_u(t) = Le^{-t} + e^{-t}e^L\frac{a^s}{a-1}(1-e^{-(a-1)t}).$$

When a is chosen such that $b \geq 1$, the expression for $z_l(t)$ is valid for $t \geq 0$. When $z_u(2\tau) > z_l(2\tau)$, one can argue that for large a, $z_u(t) > z_l(t),\ t \in [\tau,\ 2\tau]$ and $z_u(t)$ describes a lower bound for the solution starting in L for $t \in [0,\ 2\tau]$ ($z_u(t)$ reaches is maximum in $t^*(a)$ with $t^* \to 0$ as $a \to \infty$). Consequently, the trajectory with initial condition L on $[-\tau,\ 0]$, is unbounded when

$$Le^{-2\tau} + e^{-2\tau}e^L\frac{a^s}{a-1}(1-e^{-(a-1)2\tau}) > 3a\tau - \log(\frac{a^s}{a+1}).$$

□

[†] Intersection at t^* would imply $\dot{z}(t^*) > \dot{z}_e(t^*)$

5.4. Zero dynamics with eigenvalues on the imaginary axis. The situation where the zero dynamics possess eigenvalues on the imaginary axis but no eigenvalues in the open RHP deserves special attention. According to Theorem 5.1, the system is peaking, that is, the L_1 norm of the output cannot be reduced arbitrarily. However this energy can be 'spread out' over a long time interval: it is indeed well known that a system with all its eigenvalues in the closed LHP can be stabilized with a low-gain feedback, as expressed by the following theorem, taken from [13]:

THEOREM 5.7. *If a system $\dot{\xi}_0 = A_0\xi_0 + B_0 y$ is stabilizable and the eigenvalues of A_0 are in the closed left half plane, then it can be stabilized with a low-gain control law $y = K_0(a)\xi_0$ which for large a satisfies:*

$$|y(t)| \leq \frac{\gamma}{a}\|\xi_0(0)\|$$

The infinity-norm of such a low-gain control signal can be arbitrarily reduced, which results, by Theorem 4.2, in satisfactory stabilizability results when it also acts as an input disturbance of a nonlinear system. This suggests not to force the output of (5.1) exponentially fast ($\sim e^{-at}$) to zero, which results in peaking, but to drive it rapidly without peaking to the manifold $y = K_0(a)\xi_0$, on which the dynamics are controlled by the low-gain control action. Mathematically, with $e = y - K_0(a)\xi_0$ and a feedback transformation $v = u + M\xi_0$, the normal form of the ξ-subsystem is transformed into

$$\begin{array}{l} \dot{\xi}_0 = A_0\xi_0 + B_0K_0(a)\xi + B_0 e \\ e^r = v \end{array} .$$

Using a high-gain feedback driving $e(t)$ to zero without peaking, as proven in [13], proposition 4.37, one can always force the output to satisfy the constraint

$$|y(t)| \leq \gamma(e^{-at} + \frac{1}{a})\|\xi(0)\|. \tag{5.9}$$

with γ independent of a. A systematic treatment of such high-low gain control laws can be found in [8].

For instance the system,

$$\left\{ \begin{array}{l} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = u, \quad y = \xi_2 \end{array} \right. , \tag{5.10}$$

is weakly minimum-phase (zero-dynamics $\dot{\xi}_1 = 0$). With the high-low gain feedback $u = -\xi_1 - a\xi_2$ the explicit solution of (5.10) for large a can be approximated by:

$$\left[\begin{array}{c} \xi_1 \\ \xi_2 = y \end{array} \right] = c_1 e^{-at} \left[\begin{array}{c} \frac{1}{a} \\ -1 \end{array} \right] + c_2 e^{-\frac{1}{a}t} \left[\begin{array}{c} 1 \\ -\frac{1}{a} \end{array} \right]. \tag{5.11}$$

Perturbations satisfying constraint (5.9) can be decomposed in signals with vanishing L_1 and L_∞-norm. This suggests the combination of theorems 4.1 and 4.2 to:

THEOREM 5.8. *Consider the interconnected system*

$$\begin{array}{l} \dot{z} = f(z, z(t-\tau)) + \Psi(z, z(t-\tau))y \\ \dot{\xi} = A\xi + Bu \\ y = C\xi \end{array}$$

Suppose that the z-subsystem is GAS with the Lyapunov functional $V(z_t)$ satisfying Assumption 2.2 and the zeros of the ξ-subsystem are in the closed LHP. Then the *interconnected system can be made semi-globally asymptotically stable in both $[z,\ \xi]$ and the delay, using only partial-state feedback.*

Proof. As explained in Remark 4.5, the origin $(z, \xi) = (0, 0)$ is stable. Let Ω be the desired region of attraction in the $(z,\ \xi)$-space and choose R such that for all $(z_0, \xi) \in \Omega,\ \|\xi\| < R$. Because of the assumption on the ξ-subsystem, there exist partial-state feedback laws such that

$$\|y(t)\| \leq \gamma\|\xi(0)\|(e^{-at} + \frac{1}{a}) \leq \gamma R(e^{-at} + \frac{1}{a}),$$

with γ independent of a.

Consider the time-interval $[0,\ 1]$. Because

$$\int_0^1 \gamma R(e^{-at} + \frac{1}{a}) \to 0, \quad a \to \infty,$$

one can show, as in the proof of theorem 4.1, that by taking a large, the increase of V can be limited arbitrarily. Hence for $t \leq 1$, the trajectories can be bounded inside a compact region Ω_2. We can now translate the original problem over one time-unit and since

$$\sup_{t \geq 1} \gamma R(e^{-at} + \frac{1}{a}) \to 0, \quad a \to \infty,$$

we can, by Theorem 4.2, increase a until the stability domain contains Ω_2. Hence all trajectories starting in Ω are bounded and converge to the origin, because of LaSalle's theorem. □

6. Conclusions. In this paper, we first studied the effect of bounded input-perturbations on the stability of nonlinear delay equations of the form (1.1).

Global stability results are generally not possible without structural assumptions on the unperturbed system and the interconnection term, because arbitrarily small perturbations can lead to unbounded trajectories, even when they are exponentially decaying. In the ODE-case this is caused by the fact that superlinear destabilizing terms can drive the state to infinity in a finite time. Superlinear delayed terms cannot cause a finite escape-time but can still make trajectories diverge faster than any exponential function.

We also considered semi-global results when the size or shape of the perturbation can be controlled. We assumed that the unperturbed system is delay-independent stable. When the L_1 or the L_∞ norm of the perturbations is brought to zero, trajectories can be bounded semi-globally in both the state and the delay. By means of an example we explained why global results in the delay are generally not possible. Next we considered the effect of concentrating a perturbation with a fixed L_1-norm in arbitrarily small time-intervals. This leads to semi-global stabilizability in both the state and the delay (compact delay-intervals not containing $\tau = 0$), when the interconnection term is linear in its undelayed arguments.

Using these boundedness results, we studied the stabilizability of partial linear cascades (**??**) using partial state feedback. When the interconnection term is nonlinear, output peaking of the linear system can form an obstruction to semi-global stabilizability because the L_1-norm of the output cannot be reduced by achieving a faster exponential decay rate. If we assume that the interconnection term is linear in the undelayed argument and the peaking exponent is one, we have semi-global stabilizability results, because the L_1-norm of the output can be bounded from above while concentrating its energy. Even with this assumption on the interconnection term, higher peaking exponents may form an obstruction. When the zeros of the linear subsystem are in the closed left half plane, satisfactory stability results are obtained when using a high-low gain feedback, whereby the output of the linear subsystem can be decomposed in two signals with vanishing L_1 and L_∞ norm respectively.

The main contribution of this paper lies in generalizing the classical ODE results to a class of functional differential equations. Instrumental to this generalization is the observation that the way bounded input input perturbations affect a nonlinear system mainly lies in the way the L_1 and the L_∞ norm of the perturbation can be controlled.

Acknowledgements. The authors thank W.Aernouts for fruitful discussions on the results presented in the paper. This paper presents research results of the Belgian programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture (IUAP P4/02). The scientific responsibility rests with its authors.

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Appendix. Proof of Theorem 5.1. We transform the system (5.1) into the normal form:

$$\begin{aligned} \dot{\xi}_0 &= A_0\xi_0 + B_0 y \\ \dot{y} &= y_2 \\ &\vdots \\ \dot{y_r} &= u \end{aligned} \tag{A.1}$$

where $y = C[\xi_0^T\ Y^T]^T$ with $Y = [y\ y_1 \cdots y_r]^T$ is the output and $A_0 \in \mathbb{R}^{m\times m}$ represents the zero-dynamics. We consider two cases:
<u>Case 1:</u> all eigenvalues of A_0 lie in the closed RHP.
For the stabilization of the system (A.1), we use a state feedback

$$u = F_0\xi_0 + F_1 Y.$$

The closed loop matrix is:

$$A_{cl} = \left[\begin{array}{c|ccccc} A_0 & B_0 & & & & \\ \hline & 0 & 1 & & & \\ & \vdots & & \ddots & & \\ & 0 & & \cdots & 0 & 1 \\ F_0 & & & F_1 & & \end{array}\right].$$

For asymptotic stability, the observability of (F_0, A_0) is required. In the other case (unstable) eigenvalues of A_0 will still be present in the closed loop system. Mathematically, when (F_0, A_0) would not be observable, one can perform a similarity transformation on ξ_0 leading to:

$$\left[\begin{array}{c|c} A_0 & B_0 \\ \hline F_0 & \end{array}\right] \to \left[\begin{array}{cc|c} A_{\bar{o}} & A_{12} & B_{\bar{o}} \\ 0 & A_o & B_o \\ \hline 0 & F_o & \end{array}\right],$$

whereby $A_{\bar{o}}$ contains the unobservable modes of A_0. These unstable eigenvalues are still present in the closed loop matrix A_{cl}, which contradicts the stability assumption.

As a consequence the whole system is observable since the observability matrix of (C, A_{cl}) is given by

$$\mathcal{O}_{cl} = \left[\begin{array}{c|c} 0 & \mathcal{O}_{1,2} \\ \hline \mathcal{O}_{2,1} & \mathcal{O}_{2,2} \end{array}\right],$$

whereby $\mathcal{O}_{1,2}$ is the unity matrix in $\mathbb{R}^r$ and

$$\mathcal{O}_{2,1} = \begin{bmatrix} 1 & & & \\ f_n & 1 & & \\ \vdots & & \ddots & \\ f_n^{m-1} & \ldots & f_n & 1 \end{bmatrix} \cdot \begin{bmatrix} F_0 \\ F_0A_0 \\ \vdots \\ F_0A_0^{m-1} \end{bmatrix},$$

with f_n the last component of F_1. From this it follows that the observability of (C, A_{cl}) is implied by the observability of (F_0, A_0).

Consequently, in order to achieve an exponential decay of the output ($\sim e^{-at}$), we need to place *all* eigenvalues to the left of the line $\lambda = -a$. But now we are in the situation considered by Sussmann and Kokotovic[15]. From Theorem 8.1 in [15], it follows that in this case the peaking exponent equals the dimension of the zero-dynamics.

<u>Case 2:</u> A_0 has eigenvalues λ with $\mathrm{Re}(\lambda) < 0$

With another similarity transformation we split off the asymptotically stable part A_{0s} of A_0: equation (A.1) becomes:

$$\begin{cases} \dot{\xi}_{0s} = A_{0s}\xi_{0s} + A_{0su}\xi_{0u} + B_{0s}y \\ \dot{\xi}_{0u} = A_{0u}\xi_{0u} + B_{0u}y \\ y^{(r)} = u \end{cases}$$

Because ξ_{0s} is linearly coupled with the other states, it is sufficient to consider state-feedback laws for the (ξ_{0u}, Y)-subsystem (which render the eigenvalues of A_{0s} unobservable). $\square$