Boundedness properties for time-varying nonlinear systems

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Abstract

A Liapunov theorem guaranteeing uniform boundedness and uniform ultimate boundedness for a time-varying nonlinear system $\dot{x}(t) = f(x(t), t)$ has been established. The study of uniform boundedness and uniform ultimate boundedness of particular classes of time-varying nonlinear systems $\dot{x}(t) = f(x(t), t)$ is reduced to the study of the corresponding time-invariant frozen systems $\dot{x}(t) = f(x(t), \sigma)$ for all $\sigma \in \mathbb{R}$. This approach is illustrated for time-varying homogeneous systems with a positive order, for particular classes of time-varying nonhomogeneous systems and for time-varying Lotka-Volterra equations .

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1 Introduction

The stability analysis of time-varying systems $\dot{x}(t) = f(x(t), t)$ is in general more difficult than the stability analysis of time-invariant systems. For this reason, several approaches have been proposed in the literature to reduce the stability analysis of time-varying systems to the stability analysis of related time-invariant systems.

Averaging is the most popular of these techniques. Exponential stability of the (time-invariant) averaged system $\dot{x}(t) = \bar{f}(x(t))$ implies exponential stability of the original (time-varying) system provided that the time-variation of the original system is sufficiently fast [1, 2, 9]. In contrast, when the time-variation is sufficiently slow, other results have been proposed based on the stability analysis of the familiy of the frozen systems $\dot{x}(t) = f(x(t), \sigma)$ (where σ is treated as a constant parameter) [4, 9, 12, 13, 14, 15].

In the recent paper [11], it has been observed that the fast time-variation hypothesis necessary for averaging results can be replaced by a homogeneity assumption on the vector field $f(x, t)$. Because the homogeneity property affects state but not time, the time-variation of a homogeneous vector field $f(x, t)$ of positive order $\tau > 0$ is inherently fast for $\|x\|$ small and slow for $\|x\|$ large. This fast and slow variation of the vectorfield is of course to be understood relatively to the time-variation of the solutions.

Based on this observation, the main result in [11] concludes local uniform asymptotic stability of the equilibrium point $x = 0$ of the time-varying homogeneous system from asymptotic stability of the averaged system. This result exploits the inherently *fast* character of homogeneous systems with a positive order *near the origin.*

In the present paper, we exploit the inherently *slow* character of such systems *far from the origin.* We develop a freezing result for homogeneous systems with a positive order: we show that asymptotic stability of each frozen system implies uniform boundedness and uniform ultimate boundedness of the original time-varying system. Boundedness properties rather than asymptotic stability of the equilibrium point follows from the fact that the time-variation is not slow near the origin.

Our result is further extended to systems that are not necessarily homogeneous but that possess a homogeneous approximation for $\|x\|$ sufficiently large. This robustness result is dual to the robustness of local asymptotic stability with respect to higher order perturbations [7, 10].

The paper is organized as follows. In Section 2, we formulate a Liapunov result guaranteeing uniform boundedness and uniform ultimate boundedness of a time-varying system $\dot{x}(t) = f(x(t), t)$. We also explain how a Liapunov function can be constructed satisfying the conditions of this Liapunov result. This approach is used in Section 3 to prove uniform boundedness and uniform ultimate boundedness of a time-varying homogeneous system with a positive order $\tau > 0$. In Section 4, we show that the approach presented in Section 2 and Section 3 is not restricted to the class of homogeneous systems. In Section 5, we illustrate the results by means of a time-varying Lotka-Volterra system defined in the first closed orthant of $\mathbb{R}^n$.

2 Boundedness properties and the freezing technique

We first specify the class of systems under study in the present paper.

Consider

$$\dot{x}(t) = f(x(t), t) \tag{1}$$

with $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. We assume that conditions are imposed on (1) such that existence and uniqueness of its solutions is secured for all initial conditions $x_0 \in \mathbb{R}^n$ and for all initial times t_0. The solution of (1) at t with initial condition x_0 at t_0 is denoted as $x(t, t_0, x_0)$. These existence and uniqueness conditions are imposed on all the differential equations mentioned in the present paper.

We now introduce the notions of uniform boundedness and uniform ultimate boundedness (see [16], pp. 36-37).

Definition 1. The system (1) is *uniformly bounded* when [1]

- for all $R_1 > 0$, there exists a $R_2(R_1) > 0$ such that for all $x_0 \in \mathbb{R}$, for all t_0 and for all $t \geq t_0$

$$\|x_0\| \leq R_1 \Rightarrow \|x(t, t_0, x_0)\| \leq R_2(R_1). \tag{2}$$

Definition 2. The system (1) is *uniformly ultimately bounded* when

- there exists a $R > 0$ such that for all $R_1 > 0$, there exists a $T(R_1) > 0$ such that for all $x_0 \in \mathbb{R}$, for all t_0 and for all $t \geq t_0 + T(R_1)$

$$\|x_0\| \leq R_1 \Rightarrow \|x(t, t_0, x_0)\| \leq R. \tag{3}$$

The classical theorem of Liapunov proves uniform asymptotic stability of the equilibrium point $x = 0$ of a dynamical system $\dot{x}(t) = f(x(t), t)$ when there exists a positive definite and decrescent Liapunov function $V(x, t)$ whose derivative $\dot{V}(x, t)$ along the solutions of the system is negative definite. When there exists a $R_V > 0$ such that the derivative $\dot{V}(x, t)$ along the solutions of the system is negative for x with $\|x\| > R_V > 0$, the following proposition proves uniform boundedness and uniform ultimate boundedness.

Consider the system (1). Consider a continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.

Proposition 1. *Assume that for all $t \in \mathbb{R}$ and for all $x \in \mathbb{R}^n$*

$$\alpha(\|x\|) \leq V(x, t) \leq \beta(\|x\|). \tag{4}$$

The functions $\alpha(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ and $\beta(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ are class-$\mathcal{K}_\infty$ functions [2].

If there exist a class-$\mathcal{K}$ function $\gamma(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ and a $R_V > 0$ such that for all $t \in \mathbb{R}$ and for all $x \in \mathbb{R}^n$ with $\|x\| > R_V$

$$\frac{\partial V}{\partial t}(x, t) + \frac{\partial V}{\partial x}(x, t) f(x, t) \leq -\gamma(\|x\|). \tag{5}$$

then (1) *is uniformly bounded and uniformly ultimately bounded.*

Proof. The result of the present proposition has been formulated in ([16], pp. 39-42). For completeness, the proof has been included in the appendix. ☐

[1] In the present paper , we always use – without loss of generality – the Euclidean norm.

[2] A continuous function $\eta : [0, a) \to [0, \infty)$ is said to be a class-$\mathcal{K}$ function if it is stricly increasing and $\eta(0) = 0$. It is said to be a class-$\mathcal{K}_\infty$ function if $a = \infty$ and $\eta(r) \to \infty$ as $r \to \infty$.

The study uniform boundedness and uniform ultimate boundedness of a time-varying nonlinear system, by means of Proposition 1, is in general highly nontrivial. Reducing the problem to the study of time-invariant systems may be an important simplification.

Consider the time-varying system (1). For each $\sigma \in \mathbb{R}$, define the time-invariant system

$$\dot{x}(t) = f(x(t), \sigma). \tag{6}$$

We call the system (6) the frozen system of (1) at σ. Consider for each $\sigma \in \mathbb{R}$ a continuously differentiable function $V_\sigma : \mathbb{R}^n \to \mathbb{R}$. Define $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as $V(x, \sigma) := V_\sigma(x)$ for each $\sigma \in \mathbb{R}$ and each $x \in \mathbb{R}^n$.

Theorem 1. *Assume that for all $\sigma \in \mathbb{R}$ and for all $x \in \mathbb{R}^n$*

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$$W_1(x) \leq V(x, \sigma) \leq W_2(x). \tag{7}$$

  The functions $W_1 : \mathbb{R}^n \to \mathbb{R}$ and $W_2 : \mathbb{R}^n \to \mathbb{R}$ are positive definite and radially unbounded [3].

If there exists a $R_V > 0$ and positive definite functions $W_3 : \mathbb{R}^n \to \mathbb{R}$, $W_4 : \mathbb{R}^n \to \mathbb{R}$ and $W_5 : \mathbb{R}^n \to \mathbb{R}$ such that for all $\sigma \in \mathbb{R}$ and for all $x \in \mathbb{R}^n$ with $\|x\| > R_V$

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$$\left| \frac{\partial V}{\partial \sigma}(x, \sigma) \right| \leq W_3(x). \tag{8}$$

-
$$\frac{\partial V}{\partial x}(x, \sigma) f(x, \sigma) \leq -W_4(x). \tag{9}$$

-
$$W_3(x) - W_4(x) \leq -W_5(x). \tag{10}$$

then (1) *is uniformly bounded and uniformly ultimately bounded.*

Proof. The proof of the present theorem is based on Proposition 1. For each $t \in \mathbb{R}$ and for each $x \in \mathbb{R}^n$, define

$$V(x, t) := V(x, \sigma)\Big|_{\sigma = t}. \tag{11}$$

By (7) and ([9], pp. 138-139), (4) is satisfied. By (8) and (9), it is clear that for all $t \in \mathbb{R}$ and for all $x \in \mathbb{R}^n$ with $\|x\| > R_V$

$$\frac{\partial V}{\partial t}(x, t) + \frac{\partial V}{\partial x}(x, t) f(x, t) \leq W_3(x) - W_4(x) \leq -W_5(x) < 0. \tag{12}$$

By ([9], pp. 138-139), (5) is satisfied and by Proposition 1 this implies uniform boundedness and uniform ultimate boundedness for the original time-varying system (1). $\square$

Remark 1. It is obvious that the statement of Theorem 1 can be relaxed by replacing (8) and (9) by $\frac{\partial V}{\partial \sigma}(x, \sigma) + \frac{\partial V}{\partial x}(x, \sigma) f(x, \sigma) \leq -W_5(x)$. However, for the purpose we have in mind (see Section 3) the present formulation of Theorem 1 will be applied.

[3] A function $W : \mathbb{R}^n \to \mathbb{R}$ is positive definite when W is continuous, $W(0) = 0$, $W(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. In case $W(x) \to +\infty$ as $\|x\| \to +\infty$ then the positive definite function W is radially unbounded.

3 Homogeneous systems

In the present section, we specialise the result of the previous sections to the class of homogeneous systems.

Given an n-tuple $r = (r_1, ..., r_n)$ $(\forall i \in \{1, ..., n\} : r_i > 0)$. We define the dilation δ to be the map

$$\delta : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n : (s, x) \to \delta(s, x) = (s^{r_1}x_1, ..., s^{r_n}x_n) \tag{13}$$

where $x = (x_1, ..., x_2)$.

A continuous function $h : \mathbb{R}^n \to \mathbb{R}$ is r-homogeneous of degree $m \geq 0$ if and only if

$$\forall x \in \mathbb{R}^n, \forall s \in \mathbb{R}^+ : h(\delta(s, x)) = s^m h(x). \tag{14}$$

A continuous function $f_H : \mathbb{R}^n \to \mathbb{R}^n$ is r-homogeneous of order $\tau \geq 0$ if and only if

$$\forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}, \forall s \in \mathbb{R}^+ : f_H(\delta(s, x), t) = s^\tau \delta(s, f_H(x, t)). \tag{15}$$

When f_H is r-homogeneous of order $\tau \geq 0$, then for all $p > 0$, f_H is r'-homogeneous of order $\tau' \geq 0$ with $r' = (\frac{r_1}{p}, ..., \frac{r_n}{p})$ and $\tau' = \frac{\tau}{p}$. When h is r-homogeneous of degree $m \geq 0$, then for all $p > 0$, h is r'-homogeneous of degree $m' \geq 0$ with $r' = (\frac{r_1}{p}, ..., \frac{r_n}{p})$ and $m' = \frac{m}{p}$. For this reason, taking $0 < r_i < 1$ for all $i \in \{1, ..., n\}$ is not a restriction in the definition of homogeneity. In the sequel, we always take $0 < r_i < 1$ for all $i \in \{1, ..., n\}$.

A r-homogeneous norm ρ is a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}$ which is positive definite and r-homogeneous of degree 1. ($0 < r_i < 1$: $\forall i \in \{1, ..., n\}$)

In the present paper, we will use the r-homogeneous norm

$$\rho(x) = \sum_{i=1}^{n} |x_i|^{\frac{1}{r_i}}. \tag{16}$$

This homogeneous norm is continuously differentiable in $\mathbb{R}^n$ and for all $i \in \{1, ..., n\}$,

$$\frac{\partial \rho}{\partial x_i}(\delta(s, x)) = s^{1-r_i} \frac{\partial \rho}{\partial x_i}(x). \tag{17}$$

Lemma 1. *The time-invariant r-homogeneous system $\dot{x}(t) = f_H(x(t))$ of order $\tau > 0$ is uniformly asymptotically stable if and only if there exists a $k > 1$ such that for all $x_0 \in \mathbb{R}^n$, for all t_0 and for all $t \geq t_0$*

$$\rho(x(t, t_0, x_0)) \leq \frac{k\rho(x_0)}{(1 + (t - t_0)\rho(x_0)^\tau)^{\frac{1}{\tau}}}. \tag{18}$$

Proof. The proof is omitted. The reader is referred to ([5], pp. 278-284). □

3.1 Main result

Consider the r-homogeneous system

$$\dot{x}(t) = f_H(x(t), t) \tag{19}$$

with order $\tau > 0$. Consider for each $\sigma \in \mathbb{R}$ the frozen system

$$\dot{x}(t) = f_H(x(t), \sigma). \tag{20}$$

The solution of (19) at t with initial condition $x_0 \in \mathbb{R}$ at t_0 is denoted as $x_H(t, t_0, x_0)$, the solution of (20) is denoted as $x_{H\sigma}(t, t_0, x_0)$.

Theorem 2. *Assume that*

- *the equilibrium point $x = 0$ of each frozen system (20) is asymptotically stable and that the estimate (18) is uniform i.e. there exists a $k > 1$ independent of σ such that for all $\sigma \in \mathbb{R}$, for all $x_0 \in \mathbb{R}^n$, for all t_0 and for all $t \geq t_0$*

$$\rho(x_{H\sigma}(t, t_0, x_0)) \leq \frac{k\rho(x_0)}{(1 + (t - t_0)\rho(x_0)^\tau)^{\frac{1}{\tau}}}. \tag{21}$$

- *$f_H(x, \sigma)$ is continuously differentiable with respect to x and σ.*
- *There exists a $c_f > 0$ such that for all $\sigma \in \mathbb{R}$, for all $y \in \mathbb{R}^n$ with $\rho(y) = 1$ and for all $i, k \in \{1, ..., n\}$*

$$|f_{Hi}(y, \sigma)| \leq c_f \ \ \text{and} \ \ \left|\frac{\partial f_{Hi}}{\partial x_k}(y, \sigma)\right| \leq c_f \ \ \text{and} \ \ \left|\frac{\partial f_{Hi}}{\partial \sigma}(y, \sigma)\right| \leq c_f. \tag{22}$$

then the time-varying system (19) is uniformly bounded and uniformly ultimately bounded.

Proof. The proof is based on Theorem 1. Define for all $x \in \mathbb{R}^n$ and for all $\sigma \in \mathbb{R}$,

$$V(x, \sigma) := \int_0^\infty \rho(x_{H\sigma}(t, 0, x))^{m\tau} dt \tag{23}$$

where m will be chosen later on in the proof.

By (15), (22) and ([5], pp. 278-284), there exists a $k' > 0$ such that for all $\sigma \in \mathbb{R}$, for all $x \in \mathbb{R}^n$, for all t_0 and for all $t \geq t_0$

$$\rho(x_{H\sigma}(t, t_0, x)) \geq \frac{k'\rho(x)}{(1 + (t - t_0)\rho(x)^\tau)^{\frac{1}{\tau}}}. \tag{24}$$

We now prove (7), (8) and (9).

I. By (21) and (24), there exist a $c_1 > 0$ and a $c_2 > 0$ such that for all $x \in \mathbb{R}$ and for all $\sigma \in \mathbb{R}$

$$c_1\rho(x)^{(m-1)\tau} \leq V(x, \sigma) \leq c_2\rho(x)^{(m-1)\tau}. \tag{25}$$

This implies that (7) is satisfied when $m > 1$.

II. In order to verify (8), we calculate $\frac{\partial V}{\partial \sigma}(x,\sigma)$. Notice that

$$\frac{\partial V}{\partial \sigma}(x,\sigma) = m\tau \int_0^\infty \rho(x_{H\sigma}(t,0,x))^{m\tau-1} \frac{\partial}{\partial \sigma}\left(\rho(x_{H\sigma}(t,0,x)\right) dt \tag{26}$$

$$= m\tau \int_0^\infty \rho(x_{H\sigma}(t,0,x))^{m\tau-1} \left(\sum_{i=1}^n \frac{\partial \rho}{\partial x_i}(x_{H\sigma}(t,0,x)) \frac{\partial x_{H\sigma i}}{\partial \sigma}(t,0,x) \right) dt. \tag{27}$$

Here, we assume that $m > \frac{1}{\tau}$. For all $i \in \{1,...,n\}$,

$$\frac{\partial \rho}{\partial x_i}(x_{H\sigma}(t,0,x)) = \frac{\partial \rho}{\partial x_i}\left(\delta(\rho(x_{H\sigma}(t,0,x))^{-1}, x_{H\sigma}(t,0,x))\right) \rho(x_{H\sigma}(t,0,x))^{1-r_i}. \tag{28}$$

The continuity of $\frac{\partial \rho}{\partial x_i}$ on the compact set $\{y : y \in \mathbb{R}^n, \rho(y) = 1\}$ implies the existence of a $c_\rho > 0$ such that $\left|\frac{\partial \rho}{\partial x_i}(y)\right| \leq c_\rho$ for all $i \in \{1,...,n\}$ and for all $y \in \mathbb{R}^n$ with $\rho(y) = 1$. It is clear that [4]

$$\left|\frac{\partial V}{\partial \sigma}(x,\sigma)\right| \leq m\tau \int_0^\infty \rho(x_{H\sigma}(t,0,x))^{m\tau-1} c_\rho \left(\sum_{i=1}^n \rho(x_{H\sigma}(t,0,x))^{1-r_i} \left|\frac{\partial x_{H\sigma i}}{\partial \sigma}(t,0,x)\right| \right) dt. \tag{29}$$

In order to obtain an upper bound for the right hand side of (29), we first calculate an appropriate upper bound for

$$\sum_{i=1}^n \rho(x_{H\sigma}(t,0,x))^{1-r_i} \left|\frac{\partial x_{H\sigma i}}{\partial \sigma}(t,0,x)\right|. \tag{30}$$

By integrating (20), one obtains that for all $x \in \mathbb{R}^n$ and for all $t \geq 0$,

$$x_{H\sigma}(t,0,x) = x + \int_0^t f_H(x_{H\sigma}(s,0,x),\sigma) ds \tag{31}$$

and

$$\frac{\partial x_{H\sigma i}}{\partial \sigma}(t,0,x) = \int_0^t \sum_{k=1}^n \frac{\partial f_{Hi}}{\partial x_k}(x_{H\sigma}(s,0,x),\sigma) \frac{\partial x_{H\sigma k}}{\partial \sigma}(s,0,x) + \frac{\partial f_{Hi}}{\partial \sigma}(x_{H\sigma}(s,0,x),\sigma) ds. \tag{32}$$

By multiplying (32) with $\rho(x_{H\sigma}(t,0,x))^{1-r_i}$ and invoking the triangle inequality, one obtains that the expression (30) is less than or equal to

$$\sum_{i=1}^n \rho(x_{H\sigma}(t,0,x))^{1-r_i} \int_0^t \sum_{k=1}^n \left|\frac{\partial f_{Hi}}{\partial x_k}(x_{H\sigma}(s,0,x),\sigma)\right| \left|\frac{\partial x_{H\sigma k}}{\partial \sigma}(s,0,x)\right| + \left|\frac{\partial f_{Hi}}{\partial \sigma}(x_{H\sigma}(s,0,x),\sigma)\right| ds. \tag{33}$$

Notice that

$$\frac{\partial f_{Hi}}{\partial x_k}(x_{H\sigma}(s,0,x),\sigma) = \rho(x_{H\sigma}(s,0,x))^{\tau+r_i-r_k} \frac{\partial f_{Hi}}{\partial x_k}\left(\delta(\rho(x_{H\sigma}(s,0,x))^{-1}, x_{H\sigma}(s,0,x)),\sigma\right) \tag{34}$$

[4] Since $f_H(x,\sigma)$ is continuous differentiable with respect to x and σ, the solution $x_{H\sigma}(t,t_0,x_0)$ is continuously differentiable with respect to σ. ([6], Theorem 3.3, p. 21)

and

$$\frac{\partial f_{Hi}}{\partial \sigma}(x_{H\sigma}(s,0,x),\sigma) = \rho(x_{H\sigma}(s,0,x))^{\tau+r_i}\frac{\partial f_{Hi}}{\partial \sigma}\left(\delta(\rho(x_{H\sigma}(s,0,x))^{-1}, x_{H\sigma}(s,0,x)),\sigma\right). \tag{35}$$

By (22) and (33), the expression (30) is less than or equal to

$$c_f\sum_{i=1}^{n}\rho(x_{H\sigma}(t,0,x))^{1-r_i}\int_0^t\sum_{k=1}^{n}\rho(x_{H\sigma}(s,0,x))^{\tau+r_i-r_k}\left|\frac{\partial x_{H\sigma k}}{\partial \sigma}(s,0,x)\right| + \rho(x_{H\sigma}(s,0,x))^{\tau+r_i}ds. \tag{36}$$

By (21), $\rho(x_{H\sigma}(t,0,x)) \leq k\rho(x_{H\sigma}(s,0,x))$ for all $s\in[0,t]$. This implies by (36) that the expression (30) is less than or equal to

$$c_f\sum_{i=1}^{n}k^{1-r_i}\int_0^t\rho(x_{H\sigma}(s,0,x))^{1-r_i}\left(\sum_{k=1}^{n}\rho(x_{H\sigma}(s,0,x))^{\tau+r_i-r_k}\left|\frac{\partial x_{H\sigma k}}{\partial \sigma}(s,0,x)\right| + \rho(x_{H\sigma}(s,0,x))^{\tau+r_i}\right)ds. \tag{37}$$

By (21), there exists a $c_3>0$ such that for all $x\in\mathbb{R}^n$ and for all $t\geq 0$,

$$\int_0^t\rho(x_{H\sigma}(s,0,x))^{1+\tau}ds \leq c_3\rho(x). \tag{38}$$

This implies the existence of a $c_4>0$ and a $c_5>0$ such that the expression (30) is less than or equal to

$$c_4\rho(x) + c_5\int_0^t\rho(x_{H\sigma}(s,0,x)^{\tau}\left(\sum_{k=1}^{n}\rho(x_{H\sigma}(s,0,x))^{1-r_k}\left|\frac{\partial x_{H\sigma k}}{\partial \sigma}(s,0,x)\right|\right)ds. \tag{39}$$

By the Gronwall-Bellman lemma, it is clear that

$$\sum_{i=1}^{n}\rho(x_{H\sigma}(t,0,x))^{1-r_i}\left|\frac{\partial x_{H\sigma i}}{\partial \sigma}(t,0,x)\right| \leq c_4\rho(x)e^{c_5\int_0^t\rho(x_{H\sigma}(s,0,x))^{\tau}ds}. \tag{40}$$

By (21), there exists a $c_6>0$ such that for all $x\in\mathbb{R}^n$ and for all $t\geq 0$,

$$c_5\int_0^t\rho(x_{H\sigma}(s,0,x))^{\tau}ds \leq c_6\ln(1+t\rho(x)^{\tau}) \tag{41}$$

and therefore

$$\sum_{i=1}^{n}\rho(x_{H\sigma}(t,0,x))^{1-r_i}\left|\frac{\partial x_{H\sigma i}}{\partial \sigma}(t,0,x)\right| \leq c_4\rho(x)\left(1+t\rho(x)^{\tau}\right)^{c_6}. \tag{42}$$

This implies by (29) and (21) the existence of a $c_7>0$ such that

$$\left|\frac{\partial V}{\partial \sigma}(x,\sigma)\right| \leq c_7\rho(x)^{m\tau}\int_0^{\infty}\left(1+t\rho(x)^{\tau}\right)^{c_6+\frac{1}{\tau}-m}dt. \tag{43}$$

Take $m > c_6 + \frac{1}{\tau} + 1$. There exists a $c_8 > 0$ such that for all $\sigma \in \mathbb{R}$ and for all $x \in \mathbb{R}^n$,

$$\left| \frac{\partial V}{\partial \sigma}(x, \sigma) \right| \leq c_8 \rho(x)^{(m-1)\tau}. \tag{44}$$

III. From the definition (23), one obtains that the derivative of $V(x,\sigma)$ along the solutions of (20) equals $\dot{V}(x,\sigma) = -\rho(x)^{m\tau}$. This implies that

$$\frac{\partial V}{\partial x}(x,\sigma) f_H(x,\sigma) = \dot{V}(x,\sigma) = -\rho(x)^{m\tau}. \tag{45}$$

IV. By (44) and (45), (8) and (9) are satisfied with $W_3(x) = c_8\rho(x)^{(m-1)\tau}$ and $W_4(x) = \rho(x)^{m\tau}$. Since $W_3(x) - W_4(x)$ is a continuous function of x that goes to $-\infty$ as $\|x\|$ goes to $+\infty$, there exist a $R_V > 0$ and a positive definite $W_5 : \mathbb{R}^n \to \mathbb{R}$ such that for all x with $\|x\| > R_V$, $W_3(x) - W_4(x) \leq -W_5(x)$. Theorem 1 implies uniform boundedness and uniform ultimate boundedness of the system (19). □

Remark 2. By taking $V(x,\sigma)$ as defined by (23) and by setting $V(x,t) = V(x,\sigma)|_{\sigma=t}$ as in the proof of Theorem 1, we not only prove uniform boundedness and uniform ultimate boundedness of (19). We also obtain the estimate

$$\rho(x_H(t,t_0,x_0)) \leq \frac{k\rho(x_0)}{(1 + (t-t_0)\rho(x_0)^\tau)^{\frac{1}{\tau}}} \tag{46}$$

when $\|x_H(\tau,t_0,x_0)\|$ is sufficiently large for all $\tau \in [t_0,t]$. Indeed, by (44) and (45), the derivative of $V(x,t)$ along the trajectories of (19) satisfies the inequality

$$\dot{V}(x_H(t,t_0,x_0),t) \leq c_8\rho(x_H(t,t_0,x_0))^{(m-1)\tau} - \rho(x_H(t,t_0,x_0))^{m\tau}. \tag{47}$$

There exists a $R_{V2} > 0$ sufficiently large such that for all $x_H(t,t_0,x_0)$ with $\|x_H(t,t_0,x_0)\| > R_{V2}$ $\dot{V}(x_H(t,t_0,x_0),t) \leq -0.5\rho(x_H(t,t_0,x_0))^{m\tau}$. By (25), there exist positive constants c_{10} and c_{11} such that

$$\dot{V}(x_H(t,t_0,x_0),t) \leq -c_{10}V(x_H(t,t_0,x_0),t)^{\frac{m}{m-1}} \tag{48}$$

and by integration

$$V(x_H(t,t_0,x_0),t)^{\frac{1}{1-m}} \geq V(x_0,t_0)^{\frac{1}{1-m}} + c_{11}(t-t_0). \tag{49}$$

By (25), there exists a $k > 1$ such that for all t_0 and for all $t \geq t_0$ (46) is satisfied when $\|x_H(\tau,t_0,x_0)\| > R_{V2}$ for all $\tau \in [t_0,t]$.

3.2 Time-periodicity

In the present section, we consider the time-periodic case. This allows us to simplify the conditions mentioned in Theorem 2. In case the r-homogeneous system $\dot{x}(t) = f_H(x(t),t)$ with order $\tau > 0$ is time-periodic, it is possible to reformulate the first condition of Theorem 2 by only requiring asymptotic stability for each frozen system $\dot{x}(t) = f_H(x(t),\sigma)$.

Theorem 3. *Consider the system $\dot{x}(t) = f_H(x(t), t)$ where $f_H(x,t)$ is assumed to be time-periodic with period T_f. When $f_H(x,\sigma)$ is continuously differentiable with respect to x and σ and each frozen system $\dot{x}(t) = f_H(x(t),\sigma)$ is assumed to be asymptotically stable i.e., for each $\sigma \in \mathbb{R}$ there exists a $k(\sigma) > 1$ such that for all $x_0 \in \mathbb{R}^n$, for all t_0 and for all $t \geq t_0$*

$$\rho(x_{H\sigma}(t,t_0,x_0)) \leq \frac{k(\sigma)\rho(x_0)}{(1+(t-t_0)\rho(x_0)^\tau)^{\frac{1}{\tau}}}, \tag{50}$$

then the time-varying system $\dot{x}(t) = f_H(x(t),t)$ is uniformly bounded and uniformly ultimately bounded.

Proof. The proof of the present theorem is based on Theorem 2.

I. First we show that (21) is satisfied.

Since $f_H(x,\sigma)$ is continuously differentiable with respect to x and σ, $x_{H\sigma}(T,0,x)$ is continuously differentiable with respect to σ for all $T > 0$, all $x \in \mathbb{R}^n$ and all $\sigma \in [0,T_f]$ ([6], Theorem 3.3, p. 21). This implies that for all $T > 0$, all $x \in \mathbb{R}^n$ and all $\sigma \in [0,T_f]$

$$\lim_{\sigma' \to \sigma} x_{H\sigma'}(T,0,x) = x_{H\sigma}(T,0,x) \quad \text{and} \quad \lim_{\sigma' \to \sigma} \rho(x_{H\sigma'}(T,0,x)) = \rho(x_{H\sigma}(T,0,x)). \tag{51}$$

Take an arbitrary $\sigma \in [0,T_f]$. $k(\sigma)$ in (50) is not unique but for each fixed σ, the set of the possible $k(\sigma)$ has an infimum $k_{\text{inf}}(\sigma)$. We will now prove that $k_{\text{inf}}(\sigma)$ is a continuous function of σ. The continuity of $k_{\text{inf}}(\sigma)$ as a function of σ on a compact interval implies the existence of a bounded maximum $\bar{k}_{\text{inf}}$ on $[0,T_f]$. By taking an arbitrary $\epsilon > 0$ and setting $k = \bar{k}_{\text{inf}} + \epsilon$, (50) implies that (21) is satisfied.

Suppose $k_{\text{inf}}(\sigma)$ as a function of σ has a discontinuity at σ' i.e., there exists an $\epsilon' > 0$ such that for each $\delta' > 0$ there is a $\sigma'' \in]\sigma' - \delta', \sigma' + \delta'[$ for which $|k_{\text{inf}}(\sigma') - k_{\text{inf}}(\sigma'')| > \epsilon'$. This means that $k_{\text{inf}}(\sigma'') < k_{\text{inf}}(\sigma') - \epsilon'$ or that $k_{\text{inf}}(\sigma'') > k_{\text{inf}}(\sigma') + \epsilon'$.

First, suppose that $k_{\text{inf}}(\sigma'') < k_{\text{inf}}(\sigma') - \epsilon'$. For this fixed σ', $k_{\text{inf}}(\sigma')$ is the infimum of all possible $k(\sigma')$. By (50), there exists a $T' > 0$ and a $x' \in \mathbb{R}^n$ such that

$$\rho(x_{H\sigma'}(T',0,x')) > \frac{(k_{\text{inf}}(\sigma') - \frac{\epsilon'}{4})\rho(x')}{(1+T'\rho(x')^\tau)^{\frac{1}{\tau}}}. \tag{52}$$

But by (50)

$$\rho(x_{H\sigma''}(T',0,x')) \leq \frac{(k_{\text{inf}}(\sigma'') + \frac{\epsilon'}{4})\rho(x')}{(1+T'\rho(x')^\tau)^{\frac{1}{\tau}}} \leq \frac{\left(k_{\text{inf}}(\sigma') - \frac{3\epsilon'}{4}\right)\rho(x')}{(1+T'\rho(x')^\tau)^{\frac{1}{\tau}}} \tag{53}$$

such that

$$\rho(x_{H\sigma'}(T',0,x')) - \rho(x_{H\sigma''}(T',0,x')) > \frac{\epsilon'}{2}\frac{\rho(x')}{(1+T'\rho(x')^\tau)^{\frac{1}{\tau}}} \tag{54}$$

for all $\delta' > 0$ with $\sigma'' \in]\sigma' - \delta', \sigma' + \delta'[$. Since (54) contradicts with (51), the assumption that $k_{\text{inf}}(\sigma'') < k_{\text{inf}}(\sigma') - \epsilon'$ is false.

Suppose that $k_{\inf}(\sigma'') > k_{\inf}(\sigma') + \epsilon'$. For this fixed σ'', $k_{\inf}(\sigma'')$ is the infimum of all possible $k(\sigma'')$. There exists a $T'' > 0$ and a $x'' \in \mathbb{R}^n$ such that

$$\rho(x_{H\sigma''}(T'', 0, x'')) > \frac{(k_{\inf}(\sigma'') - \frac{\epsilon'}{4})\rho(x'')}{(1 + T''\rho(x'')^\tau)^{\frac{1}{\tau}}}. \tag{55}$$

But by (50),

$$\rho(x_{H\sigma'}(T'', 0, x'')) \leq \frac{(k_{\inf}(\sigma') + \frac{\epsilon'}{4})\rho(x'')}{(1 + T''\rho(x'')^\tau)^{\frac{1}{\tau}}} \leq \frac{\left(k_{\inf}(\sigma'') - \frac{3\epsilon'}{4}\right)\rho(x'')}{(1 + T''\rho(x'')^\tau)^{\frac{1}{\tau}}} \tag{56}$$

such that

$$\rho(x_{H\sigma''}(T'', 0, x'')) - \rho(x_{H\sigma'}(T'', 0, x'')) > \frac{\epsilon'}{2}\frac{\rho(x'')}{(1 + T''\rho(x'')^\tau)^{\frac{1}{\tau}}} \tag{57}$$

for all $\delta' > 0$ with $\sigma'' \in]\sigma' - \delta', \sigma' + \delta'[$. Since (57) contradicts with (51), the assumption that $k_{\inf}(\sigma'') > k_{\inf}(\sigma') + \epsilon'$ is false.

Since the discontinuity assumptions lead to contradictions, $k_{\inf}(\sigma)$ is a continuous function of σ. Therefore, $k_{\inf}(\sigma)$ has a bounded maximum $\bar{k}_{\inf}$ on $[0, T_f]$. By taking an arbitrary $\epsilon > 0$ and setting $k = \bar{k}_{\inf} + \epsilon$, (50) implies that (21) is satisfied.

II. Since $f_H(x, \sigma)$ is periodic in the second variable and since $f_H(x, \sigma)$ is continuously differentiable with respect to x and σ, (22) is satisfied.

III. Theorem 2 implies uniform boundedness and uniform ultimate boundedness of the homogeneous system $\dot{x}(t) = f_H(x(t), t)$ with order $\tau > 0$. □

4 Homogeneous approximations far from the origin

In the present section, we generalize the results of Section 3. We consider systems that have a dominant homogeneous approximation at infinity i.e. systems represented as $\dot{x}(t) = f_H(x(t), t) + g(x(t), t)$. Here $f_H : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is homogeneous with a positive order τ and $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a perturbation of f_H when $\|x\|$ is sufficiently large.

Consider $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. There exist a $R_g > 0$ and a continuous nonincreasing function $F : \mathbb{R}^+ \to \mathbb{R}$ with $\lim_{s\to\infty} F(s) = 0$ such that for all $x \in \mathbb{R}^n$ with $\rho(x) > R_g$ and for all $t \in \mathbb{R}$

$$\|\delta(\rho(x)^{-1}, g(x, t))\| \leq \rho(x)^\tau F(\rho(x)). \tag{58}$$

A typical example is the case where $g(x, t)$ is the sum of a finite number of homogeneous terms with the same dilation as $f_H(x, t)$ and with orders *smaller* than τ. For x with $\|x\|$ sufficiently large, $g(x, t)$ can be seen as a perturbation which does not affect the uniform boundedness and the uniform ultimately boundedness property.

Consider the system

$$\dot{x}(t) = f_H(x(t), t) + g(x(t), t) \tag{59}$$

and the frozen systems

$$\dot{x}(t) = f_H(x(t),\sigma) + g(x(t),\sigma). \tag{60}$$

The solution of $\dot{x}(t) = f_H(x(t),t)$ at t with initial condition $x_0 \in \mathbb{R}$ at t_0 is denoted as $x_H(t,t_0,x_0)$, the solution of $\dot{x}(t) = f_H(x(t),\sigma)$ is denoted as $x_{H\sigma}(t,t_0,x_0)$, the solution of (59) is denoted as $x(t,t_0,x_0)$,the solution of (60) is denoted as $x_\sigma(t,t_0,x_0)$.

Theorem 4. *Assume that all the conditions of Theorem 2 are satisfied, then the time-varying system $\dot{x}(t) = f_H(x(t),t) + g(x(t),t)$ is uniformly bounded and uniformly ultimately bounded.*

Proof. In order to prove the boundedness properties, consider the Liapunov function $V(x,\sigma)$ defined by (23) which satisfies (25), (44) and (45).

I By making calculations similar to the calculations in part **II** in the proof of Theorem 2 leading to (44), one obtains $\frac{\partial V}{\partial x_j}(x,\sigma)$ for all $j \in \{1,...,n\}$ and for all $x \in \mathbb{R}^n$. There exists a $c_9 > 0$ such that for all $j \in \{1,...,n\}$ and for all $x \in \mathbb{R}^n$

$$\left|\frac{\partial V}{\partial x_j}(x,\sigma)\right| \leq c_9\rho(x)^{(m-1)\tau - r_j} \tag{61}$$

when $m > c_6 + \frac{1}{\tau} + 1$.

II By (58), for all $x \in \mathbb{R}^n$ with $\rho(x) > R_g$, for all $t \in \mathbb{R}$ and for all $j \in \{1,...,n\}$

$$|g_j(x,t)| \leq \rho(x)^{r_j+\tau}F(\rho(x)). \tag{62}$$

From (45), (61) and (62)

$$\sum_{j=1}^{n}\frac{\partial V}{\partial x_j}(x,\sigma)(f_{Hj}(x,\sigma) + g_j(x,\sigma)) \leq -\rho(x)^{m\tau}(1 - nc_9F(\rho(x))) \tag{63}$$

when $\rho(x) > R_g$. Since $\lim_{s\to\infty} F(s) = 0$, there exists a $\rho_F > R_g$ such that for all $x \in \mathbb{R}^n$ with $\rho(x) > \rho_F$, $F(\rho(x)) < \frac{1}{2nc_9}$. This implies that for all $x \in \mathbb{R}^n$ with $\rho(x) > \rho_F$

$$\frac{\partial V}{\partial x}(x,\sigma)(f_H(x,\sigma) + g(x,\sigma)) < -\frac{1}{2}\rho(x)^{m\tau}. \tag{64}$$

III By (44) and (64), (8) and (9) are satisfied with $W_3(x) = c_8\rho(x)^{(m-1)\tau}$ and $W_4(x) = \frac{1}{2}\rho(x)^{m\tau}$. Since $W_3(x) - W_4(x)$ is a continuous function of x that goes to $-\infty$ as $\|x\|$ goes to $+\infty$, there exist a $R_V > 0$ and a positive definite $W_5 : \mathbb{R}^n \to \mathbb{R}$ such that for all $x \in \mathbb{R}^n$ with $\|x\| > R_V$, $W_3(x) - W_4(x) \leq -W_5(x)$. Theorem 1 implies uniform boundedness and uniform ultimate boundedness of the system (59). □

The boundedness results of Theorem 4 do not require time-periodicity of the system $\dot{x}(t) = f_H(x(t),t)$. In Theorem 5, we consider the time-periodic case which allows a simplification of the conditions.

Theorem 5. *Consider the system $\dot{x}(t) = f_H(x(t),t)$ with order $\tau > 0$. Here, $f_H(x,t)$ is assumed to be time-periodic with period T_f. When $f_H(x,\sigma)$ is continuously differentiable with respect to x and σ and each frozen system $\dot{x}(t) = f_H(x(t),\sigma)$ is assumed to be asymptotically stable then the time-varying system $\dot{x}(t) = f_H(x(t),t) + g(x(t),t)$ is uniformly bounded and uniformly ultimately bounded.*

Proof. All the conditions imposed by Theorem 3 are satisfied. The proof of Theorem 3 implies that the conditions imposed by Theorem 4 (and equivalently by Theorem 2) are satisfied. This implies uniform boundedness and uniform ultimate boundedness of the time-varying system $\dot{x}(t) = f_H(x(t), t) + g(x(t), t)$. $\square$

If we specialise the result of Theorem 5 to the case where $f_H(x, t) = f_H(x)$ is time-invariant, Theorem 5 shows that asymptotic stability of $\dot{x}(t) = f_H(x(t))$ implies uniform boundedness and uniform ultimate boundedness of

$$\dot{x}(t) = f_H(x(t)) + g(x(t), t). \tag{65}$$

The boundedness properties are determined by $f_H(x)$ and not by $g(x, t)$. By (58), for x with $\|x\|$ sufficiently large, $g(x, t)$ can be seen as a perturbation which does not affect the uniform boundedness and the uniform ultimate boundedness property.

There is a duality between these boundedness results and the results proved by Hermes ([7], Theorem 3.3), the results proved by Morin and Samson ([10], Proposition 2) and the linearization technique [9] (pp. 127-132 and pp. 147-148). Hermes [7] proves that asymptotic stability of $\dot{x}(t) = f_H(x(t))$ implies local asymptotic stability of $\dot{x}(t) = f_H(x(t)) + g(x(t))$ in case $g(x)$ is the sum of a finite number of homogeneous terms with the same dilation as $f_H(x)$ and with orders *larger* than τ. This result is valid since for x with $\|x\|$ sufficiently small $g(x)$ can be seen as a perturbation which does not affect the local asymptotic stability property.

5 Example: Lotka-Volterra equations

Theorem 5 proves uniform boundedness and uniform ultimate boundedness for systems arising as $\dot{x}(t) = f_H(x(t), t) + g(x(t), t,)$ when all the frozen systems $\dot{x}(t) = f_H(x(t), \sigma)$ of order $\tau > 0$ have an asymptotically stable equilibrium point $x = 0$. The verification of this asymptotic stability property is crucial in the application of Theorem 5. It is obvious that the verification of this asymptotic stability property becomes much easier when the frozen systems $\dot{x}(t) = f_H(x(t), \sigma)$ belong to a class of systems whose stability properties have been studied in the literature. We illustrate this by means of an example.

Consider the time-varying Lotka-Volterra system

$$\dot{x}_i(t) = x_i(t) \left((A(t)x)_i + r_i(t) \right) \tag{66}$$

where $x = (x_1, ..., x_n)^T$. Here, $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ is periodic with period T_A and $\forall i \in \{1, ..., n\}$, $r_i : \mathbb{R} \to \mathbb{R}$.

The time-varying Lotka-Volterra equation (66) is a positive system. A system is positive if its state-components are non-negative i.e., the first closed orthant of $\mathbb{R}^n$ is positively invariant. Examples of these systems are found in a variety of applied areas such as biology, chemistry, sociology [3, 8].

Although the results in the previous sections are formulated for systems defined in $\mathbb{R}^n$, they also allow the study of positive systems defined in the first orthant of $\mathbb{R}^n$.

Indeed, in case $\dot{x}(t) = f_H(x(t), t)$ is defined in the first orthant of $\mathbb{R}^n$ with the additional condition that this first closed orthant is positively invariant for the original time-varying system and for all the time-invariant frozen systems $\dot{x}(t) = f_H(x(t), \sigma)$, the results of Theorem 2 and Theorem 3 remain valid.

Suppose that $\dot{x}(t) = f_H(x(t), t) + g(x(t), t)$ is defined in the first orthant of $\mathbb{R}^n$ with the additional condition that this first closed orthant is positively invariant for the time-varying systems $\dot{x}(t) = f_H(x(t), t) + g(x(t), t)$ and $\dot{x}(t) = f_H(x(t), t)$. Suppose also that the first closed orthant of $\mathbb{R}^n$ is positively invariant for all the time-invariant frozen systems $\dot{x}(t) = f_H(x(t), \sigma) + g(x(t), \sigma)$, and $\dot{x}(t) = f_H(x(t), \sigma)$, then the results of Theorem 4 and Theorem 5 remain valid.

Example 1. *Assume that*

- *whenever*

$$x_i \left(A(\sigma)x\right)_i = \lambda(\sigma)x_i \qquad i = 1, ..., n \tag{67}$$

  holds for some σ and for some $x \neq 0$ with $x_i \geq 0$ for all $i \in \{1, ..., n\}$, then $\lambda(\sigma) < 0$.

- *$A(\sigma)$ is continuously differentiable. There exists a $c_A > 0$ such that for all $\sigma \in \mathbb{R}$*

$$\|A(\sigma)\| \leq c_A \quad \textit{and} \quad \|\dot{A}(\sigma)\| \leq c_A. \tag{68}$$

- *there exists a $c_r > 0$ such that for all $\sigma \in \mathbb{R}$ and for all $i \in \{1, ..., n\}$*

$$|r_i(\sigma)| \leq c_r. \tag{69}$$

then the time-varying system (66) *is uniformly bounded and uniformly ultimately bounded.*

Proof. By ([8], pp. 185-187), all the systems

$$\dot{x}_i(t) = x_i(t)(A(\sigma)x(t))_i \tag{70}$$

are asymptotically stable. Take an arbitrary $r \in]0, 1[$. All the systems (70) are homogeneous with respect to the dilation $(r, ..., r)$ with order $\tau = r > 0$. Take $f_H(x, t) = (f_{H1}(x, t), ..., f_{Hn}(x, t))^T$ with $f_{Hi}(x, t) = x_i(A(t)x)_i$ and take $g_i(x, t) = r_i(t)x_i$ for all x and for all t. By setting $F(s) = \frac{\sqrt{n}c_r}{s^r}$, (58) is satisfied. By the asymptotic stability property of (70), by setting $T_f = T_A$ and by (68), the conditions required by Theorem 3 and Theorem 5 are satisfied. By Theorem 5, we obtain uniform boundedness and uniform ultimate boundedness for the time-varying positive system (66). $\square$

6 Conclusions

In the present paper, we have reduced the study of uniform boundedness and uniform ultimate boundedness of a time-varying system to the study of the time-invariant frozen systems.

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A Appendix

The present appendix contains the proof of Proposition 1.

Proof. Take an arbitray $R_1 > 0$. Define $R_2(R_1) := \max\{\alpha^{-1}(\beta(R_V)), \alpha^{-1}(\beta(R_1))\}$. Take an arbitrary $x_0 \in \mathbb{R}^n$ with $\|x_0\| \leq R_1$. In order to prove (2), suppose that for some $t_1 > t_0$, $\|x(t_1, t_0, x_0)\| > R_2(R_1)$. Because of continuity of solutions and since $R_2(R_1) \geq \max\{R_1, R_V\}$, there exists a $t_1' \in [t_0, t_1[$ such that $\|x(t_1', t_0, x_0)\| = \max\{R_1, R_V\}$ and $\|x(t, t_0, x_0)\| > \max\{R_1, R_V\}$ for all $t \in]t_1', t_1]$. Since

$$V(x(t_1, t_0, x_0)) = V(x(t_1', t_0, x_0)) + \int_{t_1'}^{t_1} \dot{V}(x(t, t_0, x_0), t)dt \tag{71}$$

and by (5) $\dot{V}(x(t, t_0, x_0)) \leq -\gamma(\|x(t, t_0, x_0)\|)$ for all $t \in]t_1', t_1]$, it is clear that

$$V(x(t_1, t_0, x_0)) \leq V(x(t_1', t_0, x_0)) \leq \beta(\max\{R_1, R_V\}). \tag{72}$$

By (72) and (4), $\|x(t_1, t_0, x_0)\| \leq \max\{\alpha^{-1}(\beta(R_V)), \alpha^{-1}(\beta(R_1))\} = R_2(R_1)$. This contradicts with the assumption that $\|x(t_1, t_0, x_0)\| > R_2(R_1)$. Therefore $\|x(t, t_0, x_0)\| \leq R_2(R_1)$ for all $t \geq t_0$ and (2) follows.

In order to prove (3), take $R = \alpha^{-1}(\beta(R_V))$. Take an arbitrary $R_1 > 0$. Take an arbitrary t_0 and $x_0 \in \mathbb{R}^n$ with $\|x_0\| \leq R_1$. The solution $x(t, t_0, x_0)$ exists for all $t \geq t_0$ since by the first part of the proof $\|x(t, t_0, x_0)\| \leq \max\{\alpha^{-1}(\beta(R_V)), \alpha^{-1}(\beta(R_1))$. Define

$$T(R_1) = \max\left\{0, \frac{\beta(R_1) - \alpha(\beta^{-1}(\alpha(R)))}{\gamma(R_V)}\right\}. \tag{73}$$

Assume that for all $t_1 \in [t_0, t_0 + T(R_1)]$, $\|x(t_1, t_0, x_0)\| > \beta^{-1}(\alpha(R)) = R_V$ such that $\|x(t, t_0, x_0)\| > \beta^{-1}(\alpha(R)) = R_V$ and by (5), $\dot{V}(x(t, t_0, x_0), t) \leq -\gamma(R_V)$ for all $t \in [t_0, t_1]$. Since for all $t_1 \in [t_0, t_0 + T(R_1)]$,

$$V(x(t_1, t_0, x_0), t_1) = V(x_0, t_0) + \int_{t_0}^{t_1} \dot{V}(x(t, t_0, x_0), t)dt \leq V(x_0, t_0) - (t_1 - t_0)\gamma(R_V) \tag{74}$$

also

$$V(x(t_0 + T(R_1), t_0, x_0), t_0 + T(R_1)) \leq \beta(R_1) - T(R_1)\gamma(R_V) \leq \alpha(\beta^{-1}(\alpha(R))). \tag{75}$$

This implies by (4) that $\|x(t_0 + T(R_1), t_0, x_0)\| \leq \beta^{-1}(\alpha(R))$ which contradicts the assumption that $\|x(t_1, t_0, x_0)\| > \beta^{-1}(\alpha(R))$ for all $t_1 \in [t_0, t_0 + T(R_1)]$. Consequently, there exists a $t_1 \in [t_0, t_0 + T(R_1)]$ such that $\|x(t_1)\| \leq \beta^{-1}(\alpha(R))$. By the first part of the proof, $\|x(t)\| \leq R$ when $t \geq t_1$ and (3) follows. $\square$