Constructive Lyapunov stabilization of nonlinear cascade systems

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Abstract

We present a global stabilization procedure for nonlinear cascade and feedforward systems which extends the existing stabilization results. Our main tool is the construction of a Lyapunov function for a class of (globally stable) uncontrolled cascade systems. This construction serves as a basis for a recursive controller design for cascade and feedforward systems. We give conditions for continuous differentiability of the Lyapunov function and the resulting control law, and propose methods for their exact and approximate computation.

Key Words: Lyapunov functions, stabilization, cascade systems, recursive designs.

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1 Introduction

This paper addresses the problem of global asymptotic stabilization, by means of state feedback, of a class of nonlinear composite systems. This class includes cascade systems [8], [12], [15], [17] and feedforward systems [11], [16]. We present a unified framework in which we prove global asymptotic stability[1] for such systems. In particular, we extend the result of Saberi *et al.*[12] for partially linear cascade systems by relaxing the asymptotic stability assumption on the nonlinear subsystem and the structural constraint on the interconnection. We also prove the existence of globally asymptotically stabilizing controllers for feedforward systems under weaker assumptions than those originally made by Teel [16] and then relaxed by Mazenc and Praly [11]. These three references stimulated our work.

Our main tool is a new construction of a Lyapunov function for the cascade system

$$(\Sigma_0) \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + h(x,\xi) \qquad f(0)=0,\ h(x,0)=0 \\ \dot{\xi} & = & A\xi \qquad\qquad\qquad\quad (x,\xi) \in I\!R^n \times I\!R^m \end{array} \right.$$

where A is Hurwitz. The linearity assumption for the ξ-subsystem is made only for convenience and our results apply to any nonlinear globally asymptotically stable and locally exponentially stable subsystem $\dot{\xi} = a(\xi)$ with a known Lyapunov function. For the nonlinear subsystem $\dot{x} = f(x)$, we assume that it is stable and that a Lyapunov function $W(x)$ is known such that $L_f W(x) \leq 0$. For the interconnection term h, we assume that it is globally Lipschitz in x for any fixed ξ.

The sum of the subsystem's Lyapunov functions $W(x)$ and $\xi^T P \xi$ cannot, in general, be a Lyapunov function for the interconnected system (Σ_0). In our approach we construct a cross-term $\Psi(x,\xi)$, such that a Lyapunov function for (Σ_0) is

$$V_0(x,\xi) := W(x) + \Psi(x,\xi) + \xi^T P \xi \tag{1.1}$$

Our construction will guarantee that $V_0(x,\xi)$ is positive definite, radially unbounded, and non-increasing along the trajectories of (Σ_0).

Under some restrictions on $\dot{x} = f(x)$ we prove that $\Psi(x,\xi)$ is C^∞ and discuss the computation of $\Psi(x,\xi)$. We specify classes of problems for which a closed form of Ψ and its partial derivatives can be obtained. For more complex situations we propose approximate computations. We also present examples of systems for which the form of the stabilizing control law designed by our method is particularly simple.

Our construction of the Lyapunov function $V_0(x,\xi)$ is the main step in the global stabilization of systems with input, obtained by various augmentations of the core system (Σ_0). In the order of increasing complexity, the first augmented system is

$$(\Sigma_1) \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + h(x,\xi) + g_1(x,\xi)u \\ \dot{\xi} & = & A\xi + bu, \qquad\qquad u \in I\!R, \end{array} \right.$$

Its uncontrolled $(u=0)$ part is Σ_0. Because (Σ_1) is in the affine form $\dot{\eta} = F(\eta) + G(\eta)u$, if a Lyapunov function V_0 with $L_F V_0 \leq 0$ is known, then a feedback control which preserves

[1] In the paper (global) asymptotic stability always means (global) asymptotic stability of the origin.

global stability is $u=-L_G V$ where $G^T=[g_1^T,b^T]$. For notational convenience we let u be a scalar; if u is a vector the control law becomes $u=-(L_G V)^T$. Of course, the stability of the system (Σ_1) is not our goal because its uncontrolled part is already stable. What we are after is global *asymptotic* stability, which we will achieve under some additional assumptions.

Our further augmentation of (Σ_0) is the second system

$$(\Sigma_2)\begin{cases}\dot{x} &= f(x)+h(x,\xi)+g_2(x,\xi,y)y\\ \dot{\xi} &= A\xi+by\\ \dot{y} &= u\end{cases}$$

Now (Σ_0) plays the part of the *zero-dynamics* with respect to the output y. Because the relative degree of the system (Σ_2) is one, and a Lyapunov function for the zero-dynamics is known, we employ the feedback passivation approach [1]. In this way we generalize [12] where it was assumed that $f(x)$ is globally asymptotically stable and $h(x,\xi)\equiv 0$.

These results for (Σ_1) and (Σ_2) are the building blocks which we configure into a recursive design procedure for our third system

$$(\Sigma_3)\begin{cases} & \dot{z}=\varphi(z)+\kappa(z,x,\xi)+\rho(z,x,\xi)u\\ (\Sigma_1) & \begin{cases}\dot{x}=f(x)+h(x,\xi)+g(x,\xi)u\\ \dot{\xi}=A\xi+bu \qquad\qquad z\in I\!\!R^{n_z}.\end{cases}\end{cases}$$

First we achieve global asymptotic stability and local exponential stability of (Σ_1) with a preliminary feedback. Then we construct a Lyapunov function V_3 for (Σ_3), treating (Σ_1) with the preliminary feedback as the ξ-subsystem of (Σ_0), and the z-subsystem as the x-subsystem of (Σ_0). If with our new Lyapunov function V_3 the feedback law $-L_G V_3$ achieves local exponential stability, the above steps can be repeated with a new block z on top of (Σ_3). In this way we develop a procedure to recursively stabilize *feedforward systems* of the form

$$(\Sigma)\begin{cases}\dot{z}_p &= \phi_p(z_p)+\kappa_p(z_p,\ldots,z_1,x,\xi)+\rho_p(z_p,\ldots,z_1,x,\xi)u\\ &\vdots\\ \dot{z}_2 &= \phi_2(z_2)+\kappa_2(z_2,z_1,x,\xi)+\rho_2(z_2,z_1,x,\xi)u\\ \dot{z}_1 &= \phi_1(z_1)+\kappa_1(z_1,x,\xi)+\rho_1(z_1,x,\xi)u\\ \dot{x} &= f(x)+h(x,\xi)+g(x,\xi)u\\ \dot{\xi} &= A\xi+bu \qquad\qquad (z_1,\ldots,z_n)\in I\!\!R^{n_1}\times\ldots\times I\!\!R^{n_p}\end{cases}$$

The feedforward systems (Σ) include as a subclass the systems for which Teel [16] achieved global stabilization with his method of nested saturation functions. Teel's results have been extended recently by Mazenc and Praly [11]. Mazenc and Praly search for scalar nonlinear "weights" $l(.)$ and $k(.)$ with which they construct a Lyapunov function for (Σ_0) without a cross-term:

$$V(x,\xi)=l(W(x))+k(\xi^T P\xi) \tag{1.2}$$

When $\frac{\partial h}{\partial \xi}(x,0)\not\equiv 0$, this construction requires a preliminary change of coordinates $(\zeta,\xi)=T(x,\xi)$ which is proven to exist when the x-subsystem of (Σ_0) has the particular form

$$\dot{x}=Fx+H_1(\xi)x+H_2(\xi)+h_1(x,\xi)\xi^2+g(x,\xi)u \tag{1.3}$$

and A and F satisfy a "nonresonance condition". Then the required change of coordinates can be obtained algebraically and the computation of our cross-term $\Psi(x,\xi)$ can be avoided. However, this imposes restrictions on the choice of $l(.)$ and $k(.)$, which are eliminated with our cross-term. Our cross-term $\Psi(x,\xi)$ has also increased flexibility and robustness in forwarding and adaptive designs developed in [6, 13].

The paper is organized as follows. In Section 2, we present an explicit construction of the cross-term $\Psi(x,\xi)$ in the Lyapunov function for (Σ_0) and give conditions for the differentiability of Ψ. In Section 3 we derive stabilizing feedback laws for augmented systems (Σ_1) and (Σ_2) and analyze the conditions for global asymptotic stability of the closed-loop systems. We proceed with the recursive controller design to stabilize (Σ_3) and give explicit conditions for stabilizability of (Σ). Computational issues are considered in Section 4 which also contains two illustrative design examples.

Throughout this paper, the solution of a differential equation $\dot{\eta} = F(\eta)$ at time $s \geq 0$, with initial condition η_0 at time $s = 0$ is denoted by $\eta(s;\eta_0)$, or simply by $\eta(s)$. When we use the notation η as the initial condition, we denote the solution by $\tilde{\eta}(s;\eta)$, or simply by $\tilde{\eta}(s)$. We employ several standard concepts from stability theory (see for instance [3]). A continuous function $\gamma : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$, $\gamma \in \mathcal{K}$, if $\gamma(s)$ is strictly increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if in addition $\gamma(s) \to \infty$ as $s \to \infty$.

We say that the system $\dot{\eta} = F(\eta)$ with an equilibrium at $\eta = 0$ is *globally stable* if there exists a positive definite, radially unbounded, function $W(\eta)$ such that $W(0) = 0$ and $L_F W \leq 0$. The notation $L_F W$ stands for $\frac{\partial W}{\partial x} F$ and is recursively extended for $k = 2, 3, \ldots$ by $L_F^k W = L_F(L_F^{k-1} W)$.

If a feedback u is designed for the system (Σ), we denote the closed-loop system by (Σ, u). Finally, $\|\cdot\|$ is used for the vector 2-norm and the induced matrix norm.

2 Construction of a Lyapunov function for (Σ_0)

We first prove boundedness of the solutions of (Σ_0). Then we proceed to the construction of the cross-term $\Psi(x,\xi)$ in the Lyapunov function V_0 and analyze its differentiability properties.

2.1 Boundedness of the solutions of (Σ_0)

Lemma 1 Suppose that the system

$$(\Sigma_0) \left\{ \begin{array}{rcll} \dot{x} &=& f(x) + h(x,\xi), & f(0) = 0,\ h(x,0) = 0 \\ \dot{\xi} &=& A\xi, & (x,\xi) \in I\!R^n \times I\!R^m \end{array} \right.$$

satisfies the following assumptions:

(A1) The interconnection $h(x,\xi)$ has linear growth in x, that is

$$\|h(x,\xi)\| \leq \gamma_1(\|\xi\|) + \gamma_2(\|\xi\|)\|x\|$$

where $\gamma_1 \in \mathcal{K}$, $\gamma_2 \in \mathcal{K}$ are differentiable at the 0.

(A2) There exist positive constants c and k such that the Lyapunov function $W(x)$, which establishes global stability of $\dot{x} = f(x)$ with $L_f W \leq 0$, also satisfies

$$\|x\| > c \Rightarrow \|\frac{\partial W}{\partial x}\| \, \|x\| \leq kW(x)$$

Then along the solutions of Σ_0 with any initial condition $(x, \xi) \in I\!R^n \times I\!R^m$, the function $W(\tilde{x}(\tau))$ is bounded for all $\tau \geq 0$.

Proof: Let (x, ξ) be an arbitrary initial condition. The following sequence of inequalities directly follows from (A1) and (A2):

$$\begin{aligned}
\dot{W} &= L_f W + L_h W \\
&\leq L_h W \\
&\leq \| \frac{\partial W}{\partial x} \| \| h \| \\
&\leq \| \frac{\partial W}{\partial x} \| \, (\gamma_1(\| \tilde{\xi} \|) + \gamma_2(\| \tilde{\xi} \|) \| \tilde{x} \|)
\end{aligned}$$

By (A1), $\| \tilde{\xi}(\tau) \|$ converges to zero exponentially fast; this implies that there exist positive constants α_1, α_2, and α and functions $C_1, C_2, C \in \mathcal{K}$, such that

$$\begin{aligned}
\dot{W} &\leq \| \frac{\partial W}{\partial x} \| \, (C_1(\|\xi\|)e^{-\alpha_1 \tau} + C_2(\|\xi\|)e^{-\alpha_2 \tau} \| \tilde{x} \|) \\
&\leq \| \frac{\partial W}{\partial x} \| \| \tilde{x} \| \, C(\|\xi\|)e^{-\alpha\tau} \text{ for } \| \tilde{x} \| \geq 1
\end{aligned}$$

Using (A2), we obtain an estimate

$$\dot{W} \leq K_1(\|\xi\|)e^{-\alpha\tau} W \tag{2.1}$$

for some $K_1 \in \mathcal{K}$ and for $\|\tilde{x}\| > \max\{1, c\}$. If $\|\tilde{x}\| > \max\{1, c\}$, this estimate proves the boundedness of $W(\tilde{x}(\tau))$ because

$$W(\tilde{x}(\tau)) \leq W(x) e^{\int_0^\tau K_1(\|\xi\|) \; e^{-\alpha s} ds} \leq K(\|\xi\|) W(x) \tag{2.2}$$

for some $K \in \mathcal{K}$. On the other, if $\|\tilde{x}\| \leq \max\{1, c\}$ W is bounded by definition. □

Because W is radially unbounded, Lemma 1 proves global boundedness of the solutions of (Σ_0). If, in addition, $\dot{x} = f(x)$ is globally asymptotically stable, Lemma 1 together with the main theorem from [14] proves global asymptotic stability of (Σ_0).

Without (A1), which prevents a finite escape time of $x(t)$ due to h, the stabilization of (Σ_1) and (Σ_2) is in general not possible without further assumptions on g_1 and g_2 (see [12]). Our new assumption (A2) is weaker that the assumptions made earlier. In [12], global asymptotic stability of (Σ_0) is proven under a global *exponential* stability assumption for

$\dot{x} = f(x)$; in [10], this assumption is replaced by a quadratic lower bound on $W(x)$ and a linear upper-bound for $\|\frac{\partial W}{\partial x}\|$. In either case, the assumptions resulted in (A2) being satisfied which was in fact the only property needed for the boundedness proofs in [12] and [10]. The class of Lyapunov functions which satisfy (A2) is large and it includes all the polynomials in x, as shown in the following lemma proven in Appendix A.

Lemma 2 If $W(x)$ is a positive definite, radially unbounded, and polynomial function of x, it satisfies the assumption (A2).

2.2 The construction of the cross-term

We now construct the cross-term $\Psi(x,\xi)$ such that

$$V_0 = W(x) + \Psi(x,\xi) + \xi^T P \xi \tag{2.3}$$

is a Lyapunov function for the core system (Σ_0). The cross-term must guarantee that V_0 is nonincreasing along the solutions of (Σ_0). The time-derivative of V_0 is

$$\dot{V}_0 = L_f W + \frac{\partial W}{\partial x} h + \dot{\Psi} - \xi^T Q \xi \tag{2.4}$$

Since $W(x)$ and $\xi^T P \xi$ with $A^T P + PA = -Q$ are the subsystems Lyapunov functions, the terms $L_f W$ and $-\xi^T Q \xi$ are nonpositive. Therefore, the negativity of $\dot{V}_0$ is ensured if the derivative of the cross-term $\Psi(x,\xi)$ is

$$\dot{\Psi} = -\frac{\partial W}{\partial x} h$$

This means that Ψ is the line-integral of $\frac{\partial W}{\partial x} h$ along the solution of (Σ_0) which starts at (x,ξ):

$$\Psi(x,\xi) \triangleq \int_0^\infty \frac{\partial W}{\partial x}(\tilde{x}(s;x,\xi))\ h(\tilde{x}(s;x,\xi),\tilde{\xi}(s;\xi))\ ds \tag{2.5}$$

The following theorem shows that this integral is well-defined and that the resulting V_0 is a Lyapunov function for (Σ_0). In particular it establishes global stability of (Σ_0).

Theorem 1 If (A1) and (A2) are satisfied then:

(i) $\Psi(x,\xi)$ exists and is continuous in $\mathbb{R}^n \times \mathbb{R}^m$

(ii) $V_0(x,\xi)$ is positive definite

(iii) $V_0(x,\xi)$ is radially unbounded

Proof: **(i)** We first prove the existence of $\Psi(x,\xi)$. Arguing as in the proof of Lemma 1, we have that for each $\tau \geq 0$

$$\left|\frac{\partial W}{\partial x}(\tilde{x}(\tau))\ h(\tilde{x}(\tau),\tilde{\xi}(\tau))\right| \ \leq \ \left\|\frac{\partial W}{\partial x}(\tilde{x}(\tau))\right\| (C_1(\|\xi\|)e^{-\alpha_1\tau} + C_2(\|\xi\|)e^{-\alpha_2\tau}\ \|\ \tilde{x}(\tau)\ \|) \quad (2.6)$$

Because $W(x)$ is radially unbounded, Lemma 1 implies that $\|\ \tilde{x}(\tau)\ \|$ and $\|\ \frac{\partial W}{\partial x}(\tilde{x}(\tau))\ \|$ are bounded for all $\tau \geq 0$. From (2.6) there exists $\gamma \in \mathcal{K}$ such that

$$\left|\frac{\partial W}{\partial x}(\tilde{x}(\tau))\ h(\tilde{x}(\tau),\tilde{\xi}(\tau))\right| \leq \gamma(\|\ (x,\xi)\ \|)e^{-\alpha\tau} \quad (2.7)$$

and, hence, the integral (2.5) exists and is bounded for all bounded (x,ξ).

Next we prove the continuity of Ψ at any fixed $(\bar{x},\bar{\xi})$. Let $B(\bar{x},\delta)$ be the ball around $\bar{x}$ with radius δ. Let $(x,\xi) \in U_\delta := B(\bar{x},\delta) \times B(\bar{\xi},\delta)$ for some $\delta > 0$. For δ sufficiently small. we will show that

$$|\ \Psi(x,\xi) - \Psi(\bar{x},\bar{\xi})\ | \leq \epsilon.$$

Without loss of generality, we can choose $\delta < 1$. Using (2.7) there exists a finite time $T > 0$ such that for all $(x,\xi) \in U_1$

$$\int_T^\infty \left|\frac{\partial W}{\partial x}(\tilde{x}(s))\ h(\tilde{x}(s),\tilde{\xi}(s))\right|\ ds < \frac{\epsilon}{4}$$

It remains to show that

$$\left|\int_0^T \left(\frac{\partial W}{\partial x}(\tilde{x})\ h(\tilde{x},\tilde{\xi}) - \frac{\partial W}{\partial x}(\bar{x})\ h(\bar{x},\bar{\xi})\right)\ ds\right| < \frac{\epsilon}{2} \quad (2.8)$$

for $\|x-\bar{x}\| + \|\xi-\bar{\xi}\|$ sufficiently small, where $(\bar{x}(\tau),\bar{\xi}(\tau))$ is the solution $(\tilde{x}(\tau;\bar{x},\bar{\xi}),\tilde{\xi}(\tau;\bar{\xi}))$.

The solutions of (Σ_0) are continuous with respect to initial conditions over the finite time interval $[0,T]$ and belong to a compact set for all initial conditions in U_1. It follows that the integrand in (2.8) uniformly converges to zero when δ tends to zero. Inequality (2.8) is, therefore, satisfied for δ sufficiently small, which establishes continuity.

(ii): The function $W(\tilde{x}(\tau))$, along the solution of (Σ_0) for each initial condition (x,ξ), satisfies

$$W(\tilde{x}(\tau)) = W(x) + \int_0^\tau \dot{W}(\tilde{x}(s),\tilde{\xi}(s))ds$$

Evaluating $\dot{W}$ yields

$$W(\tilde{x}(\tau)) - \int_0^\tau \frac{\partial W}{\partial x}(\tilde{x}(s))\ h(\tilde{x}(s),\tilde{\xi}(s))ds = W(x) + \int_0^\tau \frac{\partial W}{\partial x}(\tilde{x}(s))\ f(\tilde{x}(s))ds \quad (2.9)$$

The proof of part (i) shows that the integral on the left-hand side converges as $\tau \to \infty$ and, because $W(\tilde{x}(\tau)) \geq 0$, the left hand side is bounded from below for all $\tau \geq 0$. Since the right-hand side is a nonincreasing function of τ, we conclude that the limits of both sides exist:

$$\lim_{\tau\to\infty} W(\tilde{x}(\tau)) - \int_0^\infty \frac{\partial W}{\partial x}(\tilde{x}(s))\ h(\tilde{x}(s),\tilde{\xi}(s))\ ds = W(x) + \int_0^\infty \frac{\partial W}{\partial x}(\tilde{x}(s))\ f(\tilde{x}(s))ds$$

The integral on the left hand side is $\Psi(x,\xi)$. So, as $\tau \to \infty$, the function $W(\tilde{x}(\tau))$ converges to some finite nonnegative value

$$W_\infty(x,\xi) = W(x) + \Psi(x,\xi) + \int_0^\infty \frac{\partial W}{\partial x}(\tilde{x})\ f(\tilde{x})\ ds \tag{2.10}$$

Substituting (2.10) into (2.3) we obtain V_0 as the sum of the three nonnegative terms:

$$V_0(x,\xi) = W_\infty(x,\xi) - \int_0^\infty \frac{\partial W}{\partial x}(\tilde{x})\ f(\tilde{x})\ ds + \xi^T P \xi \geq 0 \tag{2.11}$$

It follows that $V_0(x,\xi) = 0$ implies $\xi = 0$. By construction, $V_0(x,0) = W(x)$, so we conclude that

$$V_0(x,\xi) = 0 \Rightarrow (x,\xi) = (0,0) \tag{2.12}$$

Equalities (2.11) and (2.12) imply that V_0 is positive definite.

(iii): It follows immediately from (2.11) that V_0 tends to infinity when $\| \xi \|$ tends to infinity. It is therefore sufficient to prove that $\forall \xi \in I\!R^m$

$$\lim_{\|x\|\to\infty} \left(W_\infty(x,\xi) - \int_0^{+\infty} \frac{\partial W}{\partial x}(\tilde{x}(\tau)) f(\tilde{x}(\tau)) d\tau \right) = +\infty \tag{2.13}$$

Fix $\xi \in I\!R^m$ so that the class $\mathcal{K}$ functions C_1, C_2 and C defined in the proof of Lemma 1 are constants. Using the inequality (2.6), we can write for each $\tau \geq 0$

$$\begin{aligned} \dot{W} - L_f W &= L_h W \geq - \mid L_h W \mid \\ &\geq - \| \frac{\partial W}{\partial x} \| \left(C_1 e^{-\alpha_1 \tau} + C_2 e^{-\alpha_2 \tau} \| \tilde{x} \| \right) \\ &\geq - \| \frac{\partial W}{\partial x} \| \| \tilde{x} \| C e^{-\alpha \tau} - (1 - \| \tilde{x} \|) \| \frac{\partial W}{\partial x} \| C_1 e^{-\alpha_1 \tau} \end{aligned}$$

Now we examine the second term on the right hand side. If $(1 - \| \tilde{x} \|) \leq 0$ this term can be dropped without affecting the inequality. When $(1 - \| \tilde{x} \|) > 0$ we have $\|\tilde{x}\| < 1$ so the second term is bounded by $K_2 e^{-\alpha_1 \tau}$. Therefore, we can write

$$\dot{W} - L_f W \geq - \| \frac{\partial W}{\partial x} \| \| \| \tilde{x} \| C e^{-\alpha \tau} - K_2 e^{-\alpha_1 \tau} \tag{2.14}$$

Using (A2) we obtain

$$\begin{aligned} &\dot{W} \geq -K e^{-\alpha \tau} W - K_2 e^{-\alpha_1 \tau} + L_f W \qquad \text{when } \|x\| > c \\ &\dot{W} \geq -K_1 e^{-\alpha \tau} - K_2 e^{-\alpha_1 \tau} + L_f W \qquad \text{when } \|x\| \leq c \end{aligned} \tag{2.15}$$

for some positive K and K_1 which depend only on ξ.

Inequalities (2.15) yield the following lower bounds on $W(\tilde{x}(\tau))$:

$$\begin{array}{ll} \|\tilde{x}(t)\| > c \text{ for } t \in [0,\tau) & \Rightarrow W(\tilde{x}(\tau)) \geq \phi(\tau,0)W(x) + \int_0^\tau \phi(\tau,s)(-K_2 e^{-\alpha_1 s} + L_f W) ds \\ \|\tilde{x}(t)\| \leq c \text{ for } t \in [0,\tau) & \Rightarrow W(\tilde{x}(\tau)) \geq W(x) + \int_0^\tau (-K_1 e^{-\alpha s} - K_2 e^{-\alpha_1 s} + L_f W)\ ds \end{array} \tag{2.16}$$

where $\phi(\tau,s) := e^{-\frac{K}{\alpha}(e^{-\alpha s} - e^{-\alpha\tau})}$. Noting that $1 \geq \phi(\tau,s) \geq e^{-\frac{K}{\alpha}}$ for all $\tau \geq s \geq 0$, we can combine the two bounds in (2.16) to obtain for all $\tau \geq 0$

$$W(\tilde{x}(\tau)) \geq \phi(\tau,0)W(x) + \int_0^\tau (-K_1 e^{-\alpha s} - K_2 e^{-\alpha_1 s} + L_f W)\ ds \tag{2.17}$$

Hence, introducing $\kappa(\tau) := -\int_0^\tau (K_1 e^{-\alpha s} + K_2 e^{-\alpha_1 s}) ds$, which exists and is bounded for all $\tau \geq 0$, we prove that for all $\tau \geq 0$,

$$\forall \tau \geq 0 : W(\tilde{x}(\tau)) \geq e^{-\frac{K}{\alpha}} W(x) + \int_0^\tau L_f W\ ds + \kappa(\tau) \tag{2.18}$$

Subtracting from both sides of (2.18) the term $\int_0^\tau L_f W ds$ and evaluating the limit of both sides when $\tau \to \infty$, we obtain

$$W_\infty(x,\xi) - \int_0^\infty L_f W ds \geq K_3 W(x) + \kappa^\star \tag{2.19}$$

with $\kappa^\star$ finite. It is clear from the construction that $\kappa^\star$ and K_3 may depend on $\|\xi\|$ but are independent of $\|x\|$. When $\| x \| \to \infty$, the right-hand side of (2.19) tends to infinity which proves (2.13). □

Remark 1 It is clear that when $\dot{\xi} = a(\xi)$ is globally asymptotically stable and locally exponentially stable, the same results apply to cascade systems of the form

$$\begin{array}{l} \dot{x} = f(x) + h(x,\xi) \\ \dot{\xi} = a(\xi) \end{array} \tag{2.20}$$

This extension applies as well to the differentiability results of the next section.

2.3 Differentiability of the cross-term

For a wide class of systems the function Ψ is C^∞.

Theorem 2 Let the assumptions (A1), (A2) be satisfied, and, in addition assume:

(A3) The function $f(x)$ in (Σ_0) has the form

$$f(x) = \left(\begin{array}{c} f_1(x_1) \\ F_2 x_2 + f_2(x_1,x_2) \end{array} \right), \quad x = \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) \tag{2.21}$$

and $f_2(0,x_2) = 0$; the equilibrium $x_1 = 0$ of $\dot{x}_1 = f_1(x_1)$ is globally asymptotically stable, and $\dot{x}_2 = F_2 x_2$ is Lyapunov stable.

Then the function $\Psi(x,\xi)$ defined by (2.5) is continuously differentiable in $I\!R^n \times I\!R^m$.

Proof: We begin by noting that, for any $a \geq 0$, the matrix

$$\chi(\tau) := \frac{\partial \tilde{x}(\tau)}{\partial x} e^{-a\tau}$$

satisfies the linear time-varying differential equation

$$\frac{d\chi}{d\tau} = -a\chi + \left(\frac{\partial f}{\partial x} + \frac{\partial h}{\partial x}\right)\Big|_{(\tilde{x}(\tau),\tilde{\xi}(\tau))} \chi \tag{2.22}$$

with the initial condition $\chi(0) = I$. For $a = 0$, this is the variational equation of (Σ_0). Due to $f(0,x_2) = 0$ and the asymptotic stability property of $\dot{x}_1 = f_1(x_1)$, we have

$$\lim_{\tau\to\infty} \frac{\partial f_1}{\partial x_1}(\tau) = \frac{\partial f_1}{\partial x_1}\Big|_{x_1=0} := F_1, \quad \lim_{\tau\to\infty} \frac{\partial f_2}{\partial x_2} = 0$$

We rewrite (2.22) as

$$\frac{d\chi}{d\tau} = [\mathcal{A}_1(\tau) + \mathcal{B}_1(\tau)]\chi, \;\; \mathcal{A}_1(\tau) = \begin{bmatrix} -aI + F_1 & 0 \\ \frac{\partial f_2}{\partial x_1}(\tau) & -aI + F_2 \end{bmatrix} \tag{2.23}$$

where $\mathcal{B}_1(\tau) \to 0$ as $\tau \to \infty$. The constant matrices F_1 and F_2 cannot have eigenvalues with positive real parts and $\frac{\partial f_2}{\partial x_1}(\tau)$ remains bounded for all $\tau \geq 0$. Hence, (2.23) is asymptotically stable for $a > 0$, so that $\chi(\tau)$ is bounded for all $\tau \geq 0$ and converges to zero as $\tau \to \infty$. The same is true about the time-varying matrix

$$\nu(\tau) := \frac{\partial \tilde{x}(\tau)}{\partial \xi} e^{-a\tau}$$

which satisfies

$$\frac{d\nu}{d\tau} = -a\nu + \left(\frac{\partial f}{\partial x} + \frac{\partial h}{\partial x}\right)\Big|_{(\tilde{x}(\tau),\tilde{\xi}(\tau))} \nu + \frac{\partial h}{\partial \xi}\Big|_{(\tilde{x}(\tau),\tilde{\xi}(\tau))} e^{(A-aI)\tau} \tag{2.24}$$

with the initial condition $\nu(0) = 0$.

Next we prove the differentiability of $\Psi(x,\xi)$. Using the chain rule we obtain

$$\frac{\partial \Psi}{\partial x}(x,\xi) = \int_0^\infty \psi_{\tilde{x}}(\tau) \frac{\partial \tilde{x}(\tau)}{\partial x} d\tau \tag{2.25}$$

$$\frac{\partial \Psi}{\partial \xi}(x,\xi) = \int_0^\infty \left(\psi_{\tilde{x}}(\tau) \frac{\partial \tilde{x}(\tau)}{\partial \xi} + \psi_{\tilde{\xi}}(\tau)\, e^{A\tau}\right) d\tau \tag{2.26}$$

where

$$\psi_{\tilde{x}}(\tau) := \frac{\partial \psi}{\partial \tilde{x}}\Big|_{(\tilde{x}(\tau),\tilde{\xi}(\tau))} = \left(h^T \frac{\partial^2 W}{\partial x^2} + \frac{\partial W}{\partial x}\frac{\partial h}{\partial x}\right)\Big|_{(\tilde{x}(\tau),\tilde{\xi}(\tau))} \tag{2.27}$$

$$\psi_{\tilde{\xi}}(\tau) := \frac{\partial \psi}{\partial \tilde{\xi}}\Big|_{(\tilde{x}(\tau),\tilde{\xi}(\tau))} = \frac{\partial W}{\partial x}\frac{\partial h}{\partial \xi}\Big|_{(\tilde{x}(\tau),\tilde{\xi}(\tau))} \tag{2.28}$$

Since A is Hurwitz and both h and $\frac{\partial h}{\partial x}$ vanish when $\xi = 0$, there exists a constant $\alpha > 0$ such that

$$\begin{aligned} \|h(\tau)\| &\leq \gamma_3(\|(x,\xi)\|)\; e^{-\alpha\tau} \\ \|\frac{\partial h}{\partial x}(\tau)\| &\leq \gamma_4(\|(x,\xi)\|)\; e^{-\alpha\tau} \end{aligned} \tag{2.29}$$

with $\gamma_3, \gamma_4 \in \mathcal{K}_\infty$. For some $\gamma_5, \gamma_6 \in \mathcal{K}_\infty$, this yields the estimates

$$\begin{aligned} \|\psi_{\tilde{x}}(\tau)\| &\leq \gamma_5(\|(x,\xi)\|)\; e^{-\alpha\tau} \\ \|\psi_{\tilde{\xi}}(\tau)e^{A\tau}\| &\leq \gamma_6(\|(x,\xi)\|)\; e^{-\alpha\tau} \end{aligned} \tag{2.30}$$

with which we finally obtain

$$\left\|\frac{\partial \Psi}{\partial x}(x,\xi)\right\| \;\leq\; \gamma_5(\|(x,\xi)\|)\int_0^\infty \|\chi(\tau)\|\; e^{-(\alpha-a)\tau}d\tau \tag{2.31}$$

$$\left\|\frac{\partial \Psi}{\partial \xi}(x,\xi)\right\| \;\leq\; \gamma_7(\|(x,\xi)\|)\int_0^\infty (\|\nu(\tau)\|\; e^{-(\alpha-a)\tau} + e^{-\alpha\tau})\; d\tau \tag{2.32}$$

for some $\gamma_7 \in \mathcal{K}_\infty$. Since we can choose $a < \alpha$, the integrals in (2.31) and (2.32) exist and prove the existence of the partial derivatives of Ψ. The continuity of the partial derivatives can be proven along the same lines as the continuity of Ψ. □

Corollary 1 Under the assumptions (A1), (A2), and (A3), the function $\Psi(x,\xi)$ defined by (2.5) is C^∞ in $\mathbb{R}^n \times \mathbb{R}^m$.

The proof of this corollary is given in Appendix. It is an extension of the proof of Theorem 2 and shows that the function Ψ can be differentiated as many times as f and W.

Examining the variational equations in the proof of Theorem 2, we observe that their asymptotic behavior occurs in the neighborhood of the limit sets of $\dot{x} = f(x)$. On the other hand, the differentiability properties of $\Psi(x,\xi)$ are determined by this asymptotic behavior. If the limit sets of $\dot{x} = f(x)$ are equilibria, then we give a condition under which $\Psi(x,\xi)$ is a C^r function. (We recall the fact that the existence of unstable equilibria of $\dot{x} = f(x)$ away from $x = 0$ does not contradict global stability of $x = 0$.)

Corollary 2 Let (A1), (A2), be satisfied, and in addition, assume:

(A3') The limits sets of $\dot{x} = f(x)$ consist of equilibria only, and at each equilibrium the eigenvalues of the Jacobian linearization of $f(x)$ have real parts strictly smaller than $\frac{1}{r}\alpha$ where α is defined in (2.29), and r is a positive integer.

Then the function $\Psi(x,\xi)$ defined by (2.5) is C^r in $\mathbb{R}^n \times \mathbb{R}^m$.

The proof of Corollary 2 is given in Appendix. For more complex limit sets, such as limit cycles, analogous differentiability properties can be expected to hold, as illustrated by Example 3.

3 Control law design

We have invested a lot of effort in the construction of our Lyapunov function $V_0(x,\xi)$ for the core system (Σ_0) and the purpose of our construction may be questioned. We will show in this section that the core Lyapunov function $V_0(x,\xi)$ is instrumental in global stabilization of the cascade systems, and is the starting point in the new recursive design for a large class of feedforward systems.

While the construction of $V_0(x,\xi)$ may appear complex, the designs based on it are transparently simple. In fact, the simplest design due to Jacobson [4] and Jurdjevic and Quinn [7], the so-called L_GV-design, is applicable.

A well-known virtue of the L_GV-type control laws should be kept in mind: they are optimal with respect to reasonable cost functionals of the form $J=\int_0^\infty l^2+u^2dt$, and as such possess significant robustness to input uncertainties [2, 18]. This is a highly desirable, but not an easily achievable goal in nonlinear designs. While we will not elaborate this in detail, we stress that this goal is achieved by our designs for (Σ_1), (Σ_3), and, recursively, for (Σ). Each of these designs can be interpreted as the outcome of an L_GV-design with a specially constructed Lyapunov function V, a descendant of V_0.

3.1 Stabilization of (Σ_1) and (Σ_2)

The core system (Σ_0) appears in the augmented systems (Σ_1) and (Σ_2): in (Σ_1) it is the uncontrolled part, while in (Σ_2) it is the zero-dynamics part with respect to the chosen output y.

Let us start with the design for (Σ_2). In addition to global stability of the zero-dynamics (Σ_0), the relative degree of the system (Σ_2) is one. In the terminology of [12, 1] this is a relative-degree-one weakly-minimum-phase system. This means that, if a Lyapunov function for the zero-dynamics (Σ_0) is known, a passivation design can be applied to (Σ_2). The core Lyapunov function $V_0(x,\xi)$ is available from our construction and we emlpoy it to implement the feedback transformation from u to a new control v

$$u=-\frac{\partial V_0}{\partial x}g_2-\frac{\partial V_0}{\partial \xi}b+v \tag{3.1}$$

According to [1], the feedback transformation (3.1) renders the system passive from the new input v to the output y with respect to the storage function

$$V_2(x,\xi,y)=V_0(x,\xi)+\frac{1}{2}y^2 \tag{3.2}$$

In the passivation design the next step is to select v to stabilize the system (Σ_2). By closing the loop with any strictly passive system we guarantee global stability of the closed loop system. For illustrative purposes we employ the simplest strictly passive feedback $v=-y$. With this choice for v the complete control law for the system (Σ_2) is

$$u_2(x,\xi,y)=-\frac{\partial V_1}{\partial x}g_2-\frac{\partial V_1}{\partial \xi}b-y \tag{3.3}$$

The derivative $\dot{V}_2$ along the solutions of the closed-loop system (Σ_2, u_2) is

$$\dot{V}_2 \leq L_f W(x) - \xi^T Q \xi - y^2 \leq 0 \tag{3.4}$$

By LaSalle's Invariance Principle, the solutions of the closed-loop system converge to the largest invariant set where $\xi = 0$, $L_f W(x) = 0$, and $y = 0$. Inside this set $\dot{y} = 0$, so the control (3.3) satisfies $u_2(x, 0, 0) = 0$. Because $\xi = 0$ and $y = 0$, the system (Σ_2, u_2) reduces to $\dot{x} = f(x)$.

Let us return to the simpler system (Σ_1) for which $L_G V$-control is

$$u_1(x, \xi) = -\frac{\partial V_0}{\partial x}(x, \xi) g_1(x, \xi) - \frac{\partial V_0}{\partial \xi}(x, \xi) b \tag{3.5}$$

For the closed-loop system (Σ_1, u_1) the derivative $\dot{V}_0$ is

$$\dot{V}_0(x, \xi) = L_f W(x) - \xi^T Q \xi - u_1^2(x, \xi) \leq 0 \tag{3.6}$$

and the solutions converge to the largest invariant set where $\xi = 0$, $L_f W(x) = 0$, $u_1(x, 0) = 0$. Inside this set the closed-loop system again reduces to $\dot{x} = f(x)$.

With the above two designs we have not yet achieved *asymptotic* stability. For this we need to show that the largest invariant set is just the origin.

3.2 Achieving asymptotic stability

Assuming, without loss of generality, that $g_1(x, 0) = g_2(x, 0, 0)$, we denote $u_1(x, 0) = u_2(x, 0, 0)$ by $u_0(x)$. Then the condition for asymptotic stability of both (Σ_1, u_1) and (Σ_2, u_2) is that the largest invariant set of $\dot{x} = f(x)$, contained in the set

$$E = \{x \in I\!R^n \mid L_f W(x) = 0; u_0(x) = 0\}$$

must be the origin. A sufficient condition using the Lie derivatives of $W(x)$ and $u_0(x)$ is the following result of [9]:

Lemma 3 Suppose that $u_0(x)$ is C^r for some $r \geq 1$. Then the closed-loop systems (Σ_1) and (Σ_2) are globally asymptotically stable if

$$S := \{x \in I\!R^n \mid L_f^i W(x) = 0,\ L_f^j u_0(x) = 0;\ i = 1, \ldots;\ j = 0, \ldots, r\} = \{0\} \tag{3.7}$$

Altough there are many situations in which this lemma can be directly applied, a simpler asymptotic stability condition will now be given for our class of systems. It is constructive in the sense that it can be verified before the design.

Theorem 3 If

(A4) the assumption (A3) is satisfied, $W(x)$ is locally quadratic, $\frac{\partial^2 W}{\partial x^2}(0,0) = \bar{W} > 0$, and for each $x = (0, x_2)$, the following holds:

$$\frac{\partial h}{\partial \xi}(x,0) := H_0,\ g_1(x,0) := g_0, \text{and } \frac{\partial W}{\partial x}(x) = x_2^T \bar{W}_2$$

where H_0 and $\bar{W}_2$ are constant matrices, and g_0 is a constant vector;

then (Σ_1, u_1) and (Σ_2, u_2) are globally asymptotically stable provided that the span of $\{\frac{\partial}{\partial x_2}\}$ lies in the stabilizable subspace of

$$(\bar{\Sigma}_1) \begin{cases} \dot{x}_1 = F_1 x_1 + H_{01}\xi + g_{01}u \\ \dot{x}_2 = F_{21}x_1 + F_2 x_2 + H_{02}\xi + g_{02}u \\ \dot{\xi} = A\xi + bu, \end{cases} \tag{3.8}$$

where $(\bar{\Sigma}_1)$ is the Jacobian linearization of (Σ_1) at 0.

Proof : Using (3.6) and (A3), it is sufficient to consider the invariant sets of $\dot{x}_2 = F_2 x_2$ in $E' = \{(x,\xi) = (0, x_2, 0) |\ u_0(x) = 0\}$. Using (A4), we rewrite E' as

$$E' = \{(x,\xi) = (0,x_2,0)|\ -x_2^T \bar{W}_2 g_0 - \frac{\partial \Psi}{\partial x}(x,0)g_0 - \frac{\partial \Psi}{\partial \xi}(x,0)b = 0\} \tag{3.9}$$

To show that $x_2 = 0$ is the only solution of $\dot{x}_2 = F_2 x_2$ in E', we use a state decomposition $\zeta = (\zeta_u, \zeta_s)$ of $\bar{\Sigma}_1$ into its unstabilizable and stabilizable subspaces:

$$\dot{\zeta} = \bar{A}\zeta + \bar{b}u, \quad \bar{A} = \begin{pmatrix} A_u & 0 \\ A_{us} & A_s \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 0 \\ b_s \end{pmatrix}$$

Because $\frac{\partial^2 W}{\partial x^2}(0,0) > 0$, the Jacobian linearization of $\dot{x} = f(x)$ at 0 is Lyapunov stable, for otherwise $L_f W(x)$ could not be nonpositive for all x. Let $\bar{P} > 0$ satisfy $\bar{P}\bar{A} + \bar{A}^T\bar{P} \leq 0$. The control law $\bar{u} = -2\bar{b}^T\bar{P}\zeta$ results in

$$\begin{aligned} \dot{\zeta}_u &= A_u \zeta_u \\ \dot{\zeta}_s &= (A_s - 2b_s b_s^T \bar{P}_s)\zeta_s + \bar{A}_s \zeta_u, \end{aligned} \tag{3.10}$$

with $\bar{P}_s$ being the positive definite submatrix of $\bar{P}$ corresponding to A_s. Using the detectability of the pair (b_s^T, A_s^T), we conclude that $A_s - 2b_s b_s^T \bar{P}_s$ is Hurwitz and, hence, any solution of (3.10) starting in the stabilizable subspace converges to zero. In particular, because by assumption E' belongs to the stabilizable subspace of $(\bar{\Sigma}_1)$, for any initial condition in E', the solution of $(\bar{\Sigma}_1, \bar{u})$ converges to zero.

One particular $\bar{P}$ is obtained from the quadratic Lyapunov function

$$\bar{V}(x,\xi) = \frac{1}{2}x^T \bar{W} x + \bar{\Psi}(x,\xi) + \xi^T P \xi \tag{3.11}$$

where, following Theorem 1,

$$\bar{\Psi}(x,\xi) = \int_0^\infty \bar{x}^T \bar{W}\ H_0\ \bar{\xi}(s)\ ds \tag{3.12}$$

Here $(\bar{x}(s), \bar{\xi}(s))$ is the solution of the uncontrolled system $(\bar{\Sigma}_1, u = 0)$ with the initial condition (x, ξ). Hence, the corresponding $L_G V$-control is

$$\bar{u}(x,\xi) = -x^T \bar{W} g_0 - \frac{\partial \bar{\Psi}}{\partial x} g_0 - \frac{\partial \bar{\Psi}}{\partial \xi} b - 2\xi^T P b \tag{3.13}$$

It achieves convergence of $x_2(t)$ to zero for any solution of $(\bar{\Sigma}_1, \bar{u})$ starting in E'.

To complete the proof of the theorem, we compare the above control law $\bar{u}(x,\xi)$ with the control law $u_0(x) = u_1(x,0)$ given by (3.5). We will show that these two control laws are identical in the set $\{(x,\xi)|x_1 = 0, \xi = 0\}$ and, thus, in E', both are equal to 0. Because of this, any solution of (Σ_1, u_1) which is contained in E' for all t, is also a solution of $(\bar{\Sigma}_1, \bar{u})$, and therefore it converges to zero.

In E', the control law (3.13) becomes

$$\bar{u}((0,x_2),0) = -x_2^T \bar{W}_{22} g_{02} - \frac{\partial \bar{\Psi}}{\partial x}(x,0) g_0 - \frac{\partial \bar{\Psi}}{\partial \xi}(x,0) b$$

By definition, $\Psi(x,0) = \bar{\Psi}(x,0) = 0$ for each $x \in I\!R^n$ and therefore

$$\frac{\partial \Psi}{\partial x}(x,0) = \frac{\partial \bar{\Psi}}{\partial x}(x,0) = 0 \tag{3.14}$$

Next we note that, for each initial condition $(0, x_2, 0)$ and for all $s \geq 0$,

$$\tilde{x}_1(s) = \bar{x}_1(s) \equiv 0, \quad \tilde{\xi}(s) = \bar{\xi}(s) \equiv 0 \text{ and } \tilde{x}_2(s) = \bar{x}_2(s) = e^{F_2 s} x_2$$

Hence, for each initial condition $((0, x_2), 0)$ we have

$$\frac{\partial \Psi}{\partial \xi}(x,0) = \int_0^\infty \frac{\partial^2 W}{\partial x^2} \frac{\partial \tilde{x}}{\partial \xi} h(\tilde{x}(s),0)\ ds + \int_0^\infty \frac{\partial W}{\partial x} \frac{\partial h}{\partial \xi} \frac{\partial \tilde{\xi}}{\partial \xi}(\tilde{x}(s),0)\ ds$$

The first term on the right hand side is 0 because $h(x,0) = 0$. Since $\frac{\partial \tilde{\xi}}{\partial \xi}(0) = e^{As}$, the second term becomes

$$\int_0^\infty \tilde{x}_2^T W_{22} H_{02} e^{As}\ ds$$

which, using (3.12), is equal to $\frac{\partial \bar{\Psi}}{\partial \xi}((0, x_2), 0)$. Hence, the control laws $u_0(x)$ and $\bar{u}(x,\xi)$ are identical in the set E'. □

For our recursive design which we will derive next, in addition to global asymptotic stability we also need local exponential stability. This is the subject of the following corollary, whose proof is given in Appendix.

Corollary 3 If (Σ_1) satisfies assumptions (A1), (A2), (A4), and if its Jacobian linearization $(\bar{\Sigma}_1)$ is stabilizable, then, in addition to being globally asymptotically stable, (Σ_1, u_1) and (Σ_2, u_2) are also locally exponentially stable.

Our results thus significantly advance earlier work on control of cascade systems, especially the results of [11, 12]. A comparison with [11], which requires the special form (1.3), has already been made. The assumptions made in [12] that $\dot{x} = f(x)$ is asymptotically stable and $h(x, \xi) \equiv 0$ have now been removed. Our designs include the results of [12] in their full generality because the part of ξ-dynamics with modes on the imaginary axis can be lumped with the x-subsystem. As in [12], our control law and Lyapunov function can be used for backstepping through a chain of integrators. When $h(x, \xi) \equiv 0$, the control law (3.3) is exactly equal to the one proposed in [12].

3.3 A recursive design for (Σ)

The designs for (Σ_1) and (Σ_2) will now be used as initial steps in the recursive stabilization of (Σ), where the z_i-blocks, $i = 1, \dots p$, are successively added. To begin with, we stabilize the system (Σ_3) which is obtained by adding only one z-block to (Σ_1).

The closed-loop system (Σ_3, u_3) repeats the cascade structure of the core system (Σ_0): under the assumptions of Corollary 3, the system (Σ_1, u_1) is globally asymptotically stable and locally exponentially stable. Hence, it plays the part of the ξ-subsystem. For the z-subsystem with the preliminary feedback u_1 to play the part of the x-subsystem, it must satisfy (A1) and (A2), adapted here for the current situation:

(A1) There exist differentiable class $\mathcal{K}$ functions γ_9, γ_{10}, γ_{11}, and γ_{12} such that

$$\|\kappa(z, x, \xi)\| \leq \gamma_9(\|(x, \xi)\|) + \gamma_{10}(\|(x, \xi, y)\|)\|z\|$$
$$\|\rho(z, x, \xi)\| \leq \gamma_{11}(\|(x, \xi)\|) + \gamma_{12}(\|(x, \xi, y)\|)\|z\|$$

(A2) There exists a smooth, positive definite, and radially unbounded function $U(z)$, satisfying $L_\varphi U(z) \leq 0$, and positive constants k and c such that

$$\|z\| > c \Rightarrow \|\frac{\partial U}{\partial z}\| \; \|z\| \leq kU(z)$$

Applying our construction, a Lyapunov function for the system (Σ_3, u_1) is

$$V_3(z, x, \xi, y) = U(z) + \Psi_z(z, x, \xi) + V_1(x, \xi)$$

where

$$\Psi_z(z, x, \xi) \triangleq \int_0^\infty \frac{\partial U}{\partial z}(\tilde{z}) \; (\kappa(\tilde{z}, \tilde{x}, \tilde{\xi}) + \rho(\tilde{z}, \tilde{x}, \tilde{\xi}) u_1(\tilde{x}, \tilde{\xi})) \; ds \tag{3.15}$$

The tilde denotes the solution of (Σ_3, u_1) for the initial condition (z, x, ξ). The time derivative of V_3 along the trajectories of (Σ_3, u_1) is

$$\dot{V}_3 = L_\varphi U(z) + L_f W(x) - \xi^T Q \xi - u_1^2(x, \xi) \leq 0$$

To stabilize (Σ_3), we use the additional feedback

$$v = -L_{G_3} V_3,\ G_3 := \begin{pmatrix} \rho(z,x,\xi) \\ G_1(x,\xi) \end{pmatrix}$$

so that the complete control law is

$$u_3(z,x,\xi) = u_1(x,\xi) + v = -L_{G_1} V_1 - L_{G_3} V_3 \tag{3.16}$$

This control law is continuous if $\Psi_z(z,x,\xi)$ is continuously differentiable (for instance, if the system $\dot{z} = \phi(z)$ satisfies the assumption (A3) or (A3')).

The derivative $\dot{V}_3$ along the solutions of the system (Σ_3, u_3) satisfies

$$\dot{V}_3 \leq \dot{V}_3 \mid_{v=0} - v^2 \leq 0$$

and the solutions converge to the largest invariant set of (Σ_3, u_3) where $L_\varphi U(z) = 0$, $\dot{V}_3 \mid_{v=0} = 0$, $v(z,x,\xi) = 0$. Due to $v(z,x,\xi) = 0$, on this set the (x,ξ)-subsystem of (Σ_3, u_3) reduces to (Σ_1, u_1), which is globally asymptotically stable. Hence, the solutions converge to the largest invariant set of $\dot{z} = \phi(z)$ where $L_\varphi U(z) = 0$, $v(z,0,0) = 0$. If the z-subsystem satisfies the condition (A4) and the Jacobian linearization of (Σ_3) is stabilizable, then, by Corollary 3, the closed loop system (Σ_3, u_3) is globally asymptotically stable and locally exponentially stable.

Following the same steps we can recursively achieve asymptotic stabilization of the system obtained by adding another z-block on top of (Σ_3). The new added block has to satisfy (A1) and (A2). Continuing this recursive procedure, the global asymptotic stability of (Σ) is achieved under the following conditions:

Theorem 4 For global asymptotic stabilization of the feedforward system (Σ), it is sufficient that the following four conditions are jointly satisfied:

1. A is Hurwitz and the Jacobian linearization of (Σ) at 0 is stabilizable
2. The functions $h, \kappa_i, \rho_i,\ i = 1, \ldots, p$ satisfy the linear growth assumption (A1)
3. $\dot{x} = f(x),\ \dot{z}_i = \phi_i(z_i),\ i = 1, \ldots, p$ are globally stable, and their Lyapunov functions satisfy (A2)
4. Each of the subsystems x and $z_i, i = 1, \ldots, p$ satisfies (A4)

□

4 Computational issues and examples

In general, the cross-term Ψ is the solution of the partial differential equation

$$\frac{\partial \Psi}{\partial x}(f(x) + h(x,\xi)) + \frac{\partial \Psi}{\partial \xi} A\xi = -\frac{\partial W}{\partial x} h(x,\xi) \tag{4.1}$$

with the boundary condition $\Psi(x,0) = 0$. This equation is obtained by taking the time derivatives of both sides in (2.5). The control laws designed in the previous section make use of the partial derivatives of Ψ. Various numerical methods can be used to approximate Ψ and its partial derivatives. In particular, their values at a point (x,ξ) can be obtained by integration of a set of ordinary differential equations. Below we first present cases in which analytical expressions can be obtained.

4.1 Analytical expressions

To obtain a closed form solution for the line integral which defines the function Ψ we need closed form solutions $(\tilde{x}(s), \tilde{\xi}(s))$ of (Σ_0). Because $\tilde{\xi}(s) = e^{As}\xi$, an expression for $\tilde{x}(s)$ can be obtained for a number of particular cases in the form

$$\dot{x} = (F + H(\xi))x + h(\xi) \tag{4.2}$$

After the substitution of $\tilde{\xi}(s)$ into (4.2), the solution of the resulting time-varying linear differential equation can be substituted in the line integral (2.5), as illustrated by the following examples.

Example 1 For the second order system

$$\begin{aligned} \dot{x} &= h_1(\xi)x + h_2(\xi) \\ \dot{\xi} &= -a\xi \end{aligned} \tag{4.3}$$

we select $W(x) = x^2$, which yields the cross-term

$$\Psi(x,\xi) = \int_0^\infty 2\tilde{x}(s)(h_1(\tilde{\xi}(s))\tilde{x}(s) + h_2(\tilde{\xi}(s))\; ds \tag{4.4}$$

Substituting the solution of (4.3),

$$\tilde{x}(s) = e^{\int_0^s h_1(\tilde{\xi}(\mu))d\mu}x + \int_0^s e^{\int_\tau^s h_1(\tilde{\xi}(\mu))d\mu} h_2(\tilde{\xi}(\tau))d\tau, \quad \tilde{\xi}(s) = e^{-as}\xi$$

in the integral (4.4), the expression for $\Psi(x,\xi)$ can be written as

$$\begin{aligned} \Psi(x,\xi) &= x^2 \int_0^\infty \frac{d}{ds}\left\{ e^{2\int_0^s h_1(\tilde{\xi}(\mu))d\mu} \right\} ds \\ &\quad + 2x \int_0^\infty \frac{d}{ds}\left\{ e^{\int_0^s h_1(\tilde{\xi}(\mu))d\mu} \int_0^s e^{\int_\tau^s h_1(\tilde{\xi}(\mu))d\mu} h_2(\tilde{\xi}(\tau))d\tau \right\} ds \\ &\quad + \int_0^\infty \frac{d}{ds}\left\{ \int_0^s e^{\int_\tau^s h_1(\tilde{\xi}(\mu))d\mu} h_2(\tilde{\xi}(\tau))d\tau \right\}^2 ds \\ &= -x^2 + \left(x e^{\int_0^\infty h_1(\tilde{\xi}(\mu))d\mu} + \int_0^\infty e^{\int_\tau^\infty h_1(\tilde{\xi}(\mu))d\mu} h_2(\tilde{\xi}(\tau))d\tau \right)^2 \end{aligned}$$

Because the function $h(x,\xi) \triangleq h_1(\xi)x + h_2(\xi)$ vanishes at $\xi = 0$ we can write $h_1(\xi) = \bar{h}_1(\xi)\xi$ and $h_2(\xi) = \bar{h}_2(\xi)\xi$. Using these expressions and $\sigma = \tilde{\xi}(\mu) = \xi e^{-a\mu}$ and $u = \tilde{\xi}(\tau) = \xi e^{-a\tau}$ we obtain

$$\Psi(x,\xi) = -x^2 + \left(x e^{\frac{1}{a}\int_0^\xi \bar{h}_1(\sigma)d\sigma} + \frac{1}{a}\int_0^\xi e^{\frac{1}{a}\int_0^u \bar{h}_1(\sigma)d\sigma} \bar{h}_2(u)du \right)^2 \tag{4.5}$$

Finally a Lyapunov function for the system (4.3) is given by

$$V_0(x,\xi) = W(x) + \Psi(x,\xi) + \xi^2 = \left(x e^{\frac{1}{a}\int_0^\xi \bar{h}_1(\sigma)d\sigma} + \frac{1}{a}\int_0^\xi e^{\frac{1}{a}\int_0^u \bar{h}_1(\sigma)d\sigma} \bar{h}_2(u)du \right)^2 + \xi^2 \tag{4.6}$$

The above integrals can be explicitly solved for certain functions h_1 and h_2, or else they can be approximated. For this example, the Lyapunov construction based on the exact change of coordinates [11] gives the same function $V_0(x,\xi)$ as in (4.6). □

Example 2 When in the system below the interconnection term $p(\xi)$ is a polynomial,

$$\begin{aligned} \dot{x} &= Fx + p(\xi) \\ \dot{\xi} &= A\xi \end{aligned} \tag{4.7}$$

then the closed form solution for Ψ is also a polynomial. In particular, if p is linear then Ψ is a quadratic form.

For the sake of illustration, in the case when x and ξ are scalars and p is a quadratic polynomial, the cross-term is

$$\Psi(x,\xi) = a_1 x\xi + a_2 x\xi^2 + a_3\xi^2 + a_4\xi^3 + a_5\xi^4$$

where the coefficients are independent of x and ξ. □

If the x-dynamics of (Σ_0) is not in the form (4.2), then it is usually not possible to obtain a closed form solution for $\tilde{x}(s)$ and in turn for Ψ. Nevertheless, the next example illustrates a situation where a closed form solution for Ψ does not require the solution of the x-equation.

Example 3 Consider the system

$$\begin{aligned} \dot{x} &= F(x)x + h(\xi)x \\ \dot{\xi} &= A\xi \end{aligned} \tag{4.8}$$

where $h(\xi)$ is a scalar function and the matrix $F(x)$ satisfies $F^T(x)P + P^T F(x) \equiv 0$ for some constant positive definite matrix P. Then $W(x) = x^T P x$ satisfies $\dot{W}(x) = h(\xi)W(x)$ and, therefore,

$$W(\tilde{x}(\tau)) = W(x) e^{\int_0^\tau h(\tilde{\xi}(s))ds}. \tag{4.9}$$

On the other hand, we have

$$\Psi(x,\xi) = \int_0^\infty 2\tilde{x}^T P h(\tilde{\xi})\, \tilde{x}\, d\tau = \int_0^\infty W(\tilde{x}) h(\tilde{\xi})\, d\tau \tag{4.10}$$

Substituting (4.9) in (4.10) we obtain

$$\Psi(x,\xi) = W(x)\int_0^\infty e^{\int_0^\tau h(\tilde{\xi})ds} h(\tilde{x})d\tau = W(x)\left(e^{\int_0^\infty h(\tilde{\xi}(s))ds} - 1 \right)$$

We remark that Ψ is C^∞ although assumptions (A3) and (A3') may not be satisfied.

4.2 Evaluation of $\frac{\partial \Psi}{\partial x}$ and $\frac{\partial \Psi}{\partial \xi}$ by integration

For on-line computation when x and ξ are known at time t, we need to evaluate $\frac{\partial \Psi}{\partial x}$ and $\frac{\partial \Psi}{\partial \xi}$ with a desired accuracy.

Denote by $\Psi^\star(x,\xi,\tau)$ the line integral evaluated up to the time τ; i.e.

$$\Psi^\star(x,\xi,\tau) \triangleq \int_0^\tau \frac{\partial W}{\partial x}(\tilde{x})\ h(\tilde{x},\tilde{\xi})ds$$

We write $\Psi^\star$ as a function of τ only, but we keep in mind that it also depends on x and ξ. $\Psi^\star$ is the solution of the ordinary differential equation

$$(\Psi^\star)'(\tau) = \frac{\partial W}{\partial x}h\bigg|_{(\tilde{x}(\tau),\tilde{\xi}(\tau))}, \quad \Psi^\star(0) = 0 \tag{4.11}$$

where the notation $(\Psi^*)'$ stands for $\frac{d\Psi^*}{d\tau}$. By taking the partial derivatives with respect to initial conditions x and ξ, we obtain the following differential equations (in the notation as of (2.27) and (2.28))

$$\left(\frac{\partial \Psi^\star}{\partial x}\right)'(\tau) = \psi_{\tilde{x}}(\tau)\,\chi(\tau)\,e^{a\tau} \tag{4.12}$$

$$\left(\frac{\partial \Psi^\star}{\partial \xi}\right)'(\tau) = \psi_{\tilde{x}}(\tau)\,\nu(\tau)\,e^{a\tau} + \psi_{\tilde{\xi}}(\tau)e^{A\tau} \tag{4.13}$$

with the initial conditions $\frac{\partial \Psi^\star}{\partial x}(0) = 0$ and $\frac{\partial \Psi^\star}{\partial \xi}(0) = 0$. The proof of Theorem 2 provides the bound

$$\|\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^\star}{\partial x}(T)\| \le M(\|(x,\xi)\|)\int_T^\infty e^{-(\alpha-a)s}ds = \frac{1}{\alpha-a}M(\|(x,\xi)\|)e^{-(\alpha-a)T}$$

for some $M \in \mathcal{K}_\infty$. The same bound can be established for the difference $\|\frac{\partial \Psi}{\partial \xi} - \frac{\partial \Psi^\star}{\partial \xi}(T)\|$. We summarize this as follows:

Corollary 4 Let $(x,\xi) \in \Omega$. For any given $\varepsilon > 0$ and any compact set $\Omega \subset R^{n+m}$, there exists a constant $T > 0$ such that

$$\|\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^\star}{\partial x}(\tau)\| < \varepsilon \tag{4.14}$$

$$\|\frac{\partial \Psi}{\partial \xi} - \frac{\partial \Psi^\star}{\partial \xi}(\tau)\| < \varepsilon \tag{4.15}$$

for every $\tau > T$. $\square$

In other words, to obtain the partial derivatives with the desired accuracy we have to integrate the set of equations (2.22), (2.24), (4.12), and (4.13) on an interval of sufficient length T. In general to achieve the accuracy as in Corollary 4, the interval T has to increase with the size of the compact set Ω. For on-line computation, the integrals must be evaluated in a time scale faster than real time.

4.3 Design examples

Example 4 We apply our design to the system

$$\begin{aligned} \dot{x} &= \frac{x^2}{1+x^2}\xi - x^2\xi u \\ \xi &= -\xi + u \end{aligned} \tag{4.16}$$

which is of the form Σ_1. The control law

$$u_1(x,\xi) = \frac{\partial V_1}{\partial x}x^2\xi - \frac{\partial V_1}{\partial \xi}$$

requires the construction of a Lyapunov function for the uncontrolled system which is in the form (Σ_0):

$$\begin{aligned} \dot{x} &= \frac{x^2}{1+x^2}\xi \\ \dot{\xi} &= -\xi \end{aligned} \tag{4.17}$$

Assumption (A1) is satisfied because

$$|h(x,\xi)| = \left|\frac{x^2}{1+x^2}\xi\right| \leq |\xi| \;\forall x$$

Since $f(x) \equiv 0$, Assumption (A2) is satisfied with $W(x) = x^2$. The cross term Ψ is

$$\Psi(x,\xi) = \int_0^\infty \frac{\partial W}{\partial x} h \, ds = \int_0^\infty \frac{\partial W}{\partial x}(\tilde{x})\dot{\tilde{x}} \, ds = \int_0^\infty dW = \lim_{s\to\infty} W(\tilde{x}(s)) - W(x)$$

and our construction yields the Lyapunov function

$$V_0(x,\xi) = W(x) + \Psi(x,\xi) + \xi^2 = \lim_{s\to\infty} W(\tilde{x}(s)) + \xi^2 = W(x_\infty) + \xi^2 \tag{4.18}$$

where $x_\infty := \lim_{s\to\infty}\tilde{x}(s)$. The differential equation for x is simple enough to be solved analytically: $\tilde{x}(s,(x,\xi)) \equiv 0$ for $x = 0$ and

$$\tilde{x}(s,(x,\xi)) = \frac{x^2 - 1 + x\xi(1-e^{-s}) + \sqrt{(x^2-1+x\xi(1-e^{-s}))^2 + 4x^2}}{2x} \quad \text{if} \quad x \neq 0$$

Taking the limit for $s \to \infty$, we set

$$x_\infty(x,\xi) = \frac{x^2 - 1 + x\xi + \sqrt{(x^2-1+x\xi)^2 + 4x^2}}{2x} \quad \text{if} \quad x \neq 0 \tag{4.19}$$

Substituting (4.19) into (4.18) yields

$$V_0(x,\xi) = \begin{cases} \dfrac{\left(x^2-1+x\xi+\sqrt{(x^2-1+x\xi)^2+4x^2}\right)^2}{4x^2} + \xi^2 & \text{if} \quad x \neq 0 \\ \xi^2 & \text{if} \quad x = 0 \end{cases} \tag{4.20}$$

Assumption (A3) is satisfied with $x = x_2$ and $F_2 = 0$. From Theorem 2 and Corollary 1, we conclude that V_0 is C^∞.

To derive the control law $u_1(x,\xi)$ we compute

$$\frac{\partial V_0}{\partial x} = 2\frac{x^2+1}{\sqrt{(x^2-1+x\xi)^2+4x^2}}\frac{x_\infty^2}{x}, \quad \frac{\partial V_0}{\partial \xi} = 2x_\infty^2 x + 2\xi$$

and obtain

$$u_1(x,\xi) = -2\frac{x^2+1}{\sqrt{(x^2-1+x\xi)^2+4x^2}}\xi x x_\infty^2 - 2x x_\infty^2 - 2\xi$$

Because $u_1(x,0) = -2x_\infty^2(x,0)x = -2x^3$, the stabilizability condition (3.7) is satisfied and the closed loop system is globally asymptotically stable. We note that the x-subsystem of (4.17) is not in the special form (1.3), so that the method of [11] cannot be applied directly. $\square$

Example 5 To illustrate our design procedure, we use the model of a mechanical system, for which other control laws have been proposed in [5], [19]. After a change of coordinates and a feedback transformation this model takes the form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 + \sin z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= u \end{aligned} \tag{4.21}$$

A further change of coordinates $x_1 = z_1, x_2 = z_2, \xi = z_3$, and $y = z_3 + z_4$ transforms the system into the form (Σ_2):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \sin\xi \\ \dot{\xi} &= -\xi + y \\ \dot{y} &= -\xi + y + u \end{aligned} \tag{4.22}$$

The third-order zero dynamics subsystem (x_1, x_2, ξ) is stable. By setting $W(x) = x^T x$ we obtain the cross term

$$\Psi(x,\xi) = -x^T x + (x+\beta(\xi))^T(x+\beta(\xi))$$

where

$$\beta(\xi) = \begin{bmatrix} \beta_1(\xi) \\ \beta_2(\xi) \end{bmatrix} = \int_0^\infty \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix} \begin{bmatrix} 0 \\ \sin(\xi e^{-s}) \end{bmatrix} ds = \int_0^\infty \begin{bmatrix} -\sin s \sin(\xi e^{-s}) \\ \cos s \sin(\xi e^{-s}) \end{bmatrix} ds$$

The formula (3.3) for the control law yields

$$u = -2y - 2(x_1 + \beta_1)\frac{\partial \beta_1}{\partial \xi} - 2(x_2 + \beta_2)\frac{\partial \beta_2}{\partial \xi} \tag{4.23}$$

and because the conditions of Theorem 3 are satisfied the closed loop system is globally asymptotically stable.

The integrals defining functions β_1 and β_2 cannot be evaluated in the closed form. Nevertheless, β_1 and β_2 are very nicely behaving analytic functions and a number of different approximation methods can be used for the control implementation. For example, the Taylor expansion gives

$$\beta_1(\xi) = \sum_0^\infty \frac{(-1)^{k+1}}{((2k+1)^2+1)(2k+1)!}\xi^{2k+1} = -\frac{1}{2}\xi + \frac{1}{10\ 3!}\xi^3 - \frac{1}{26\ 5!}\xi^5 + \frac{1}{50\ 7!}\xi^7 + \dots$$

$$\beta_2(\xi) = \sum_0^\infty \frac{(-1)^{k}}{((2k+1)^2+1)(2k)!}\xi^{2k+1} = -\xi\frac{d\beta_1}{d\xi}$$

but is not the most efficient implementation. Another implementation method is to evaluate the required integrals for β, $\frac{d\beta_1}{d\xi}$, $\frac{d\beta_2}{d\xi}$ on-line at current value of ξ and thus avoid precomputing and storing the data. This on-line method seems promising in many situations.

5 Conclusion

This paper presents a systematic, recursive design procedure for global stabilization of cascade and feedforward systems. Our design relies on the new construction of a Lyapunov function for (Σ_0) which is a cascade of a stable (but not necessarily asymptotically stable) system and an exponentially stable system. The Lyapunov function for the cascade is obtained by adding to the sum of the original Lyapunov functions a cross-term Ψ which takes into account the effects of the interconnection.

Continuous differentiability of Ψ established in Section 2 is required for continuity of the designed feedback laws. We have shown by Example 3 that our sufficient conditions are not necessary and the differentiability of Ψ is an unresolved issue. We have designed feedback laws and given conditions for global asymptotic stability for systems augmented from (Σ_0): for (Σ_1), (Σ_0) is the uncontrolled system, and for (Σ_2), (Σ_0) is the zero-dynamics. The stabilization method for (Σ_1) serves as a basis for a recursive stabilization of (Σ). Each step of the recursion involves the construction of a new Lyapunov function, including the construction of a new cross-term, for a cascade system which is globally stable. Computational aspects for Ψ and in turn for the control laws have been discussed in Section 4. In particular, various situations have been presented where an exact and explicit solution can be obtained. Our construction always gives a control laws which can be computed pointwise.

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A Appendix

A.1 Proof of Lemma 2

Choose $k := 4N^*$ where N^* is the degree of the polynomial $W(x)$. Define $S(0,1) := \{x \in I\!R^n \mid \|x\| = 1\}$ and pick any $y \in S(0,1)$. First we show that there exists $c(y)$ such that

$$\lambda \| \frac{\partial W}{\partial x}(\lambda y) \| < kW(\lambda y) \ \text{ for } \lambda \geq c(y) > 0 \tag{A.1}$$

First assume that $y = e_1$ where $e_1 = (1, 0, \ldots, 0)^T$. Then $W(\lambda y) = P(\lambda)$ with P a polynomial in λ. Let $a_N \lambda^N$ be the highest-order term of P (clearly $N \leq N^*$). Because a_N must be positive, for λ sufficiently large,

$$W(\lambda y) = P(\lambda) > \frac{a_N \lambda^N}{2} \tag{A.2}$$

$$\| \frac{\partial W}{\partial x}(\lambda y) \| = \mid P'(\lambda) \mid < 2 N a_N \lambda^{N-1} \tag{A.3}$$

From (A.2) and (A.3) follows

$$\lambda \| \frac{\partial W}{\partial x}(\lambda y) \| \leq 2 N a_N \lambda^N < 4N \ W(\lambda y) \tag{A.4}$$

which establishes (A.1) for $y = e_1$ since $4N \leq k$.

For $y \in S(0,1)$ arbitrary, there exists an orthonormal matrix T such that $y = Te_1$. Defining $z = T^{-1}x$, we obtain a new polynomial Q in z:

$$Q(z) = W(x) = W(Tz)$$

Due to linearity of the transformation, Q is positive definite, radially unbounded, and polynomial function of z of order N^*. Moreover,

$$\|\frac{\partial W}{\partial x}(x)\| \leq \|\frac{\partial Q}{\partial z}(T^{-1}x)\|\|T^{-1}\| = \|\frac{\partial Q}{\partial z}(z)\|$$

In particular, we obtain for $x = \lambda y$

$$W(\lambda y) = W(\lambda T e_1) = Q(\lambda e_1)$$
$$\|\frac{\partial W}{\partial x}(\lambda y)\| \leq \|\frac{\partial Q}{\partial z}(\lambda e_1)\|$$

Since the inequality (A.4) applies to $Q(\lambda e_1)$, we conclude that

$$\lambda\|\frac{\partial W}{\partial x}(\lambda y)\| \leq \lambda\|\frac{\partial Q}{\partial z}(\lambda e_1)\| < kQ(\lambda e_1) = kW(\lambda y)$$

for $\lambda > c(y)$, which establishes (A.1) for y arbitrary.

Since W and $\frac{\partial W}{\partial x}$ are continuous and the inequality (A.1) is strict, each $y \in S(0,1)$ has an open neighborhood $\mathcal{O}(y)$ in $S(0,1)$ such that

$$x \in \mathcal{O}(y) \Rightarrow \lambda\|\frac{\partial W}{\partial x}(\lambda x)\| < kW(\lambda x) \text{ for } \lambda \geq c(y)$$

The union of the neighborhoods $(\mathcal{O}(y))_{y\in S(0,1)}$ provides an open covering of $S(0,1)$. By compactness of the unit sphere, there exists a finite number of points $(y_i)_{i\in I} \subset S(0,1)$ such that $U_{i\in I}\mathcal{O}(y_i)$ is still an open covering of $S(0,1)$. As a consequence, the constant c in Assumption (A2) can be chosen as the maximum of $c(y_i)$, $i \in I$. This ends the proof of Lemma 1. □

A.2 Proof of Corollary 1

We use the same notation as in the proof of Theorem 2. We only show the existence and continuity of

$$\frac{\partial^2 \Psi}{\partial x_i \partial x_j},\ 1 \leq i \leq n,\ 1 \leq j \leq n$$

Existence and continuity of partial derivatives of any order follow by induction.

Using the chain rule, we have

$$\frac{\partial^2 \Psi}{\partial x_i \partial x_j} = \int_0^\infty \left(\frac{\partial \tilde{x}}{\partial x_i}(\tau)\right)^T \frac{\partial \psi_{\tilde{x}}^T}{\partial \tilde{x}}(\tau)\,\frac{\partial \tilde{x}}{\partial x_j}(\tau)\ d\tau + \int_0^\infty \psi_{\tilde{x}}(\tau)\frac{\partial^2 \tilde{x}}{\partial x_j \partial x_i}(\tau)\ d\tau \tag{A.5}$$

From the proof of Theorem 2 we know that

$$\|\frac{\partial \psi_{\tilde{x}}^T}{\partial \tilde{x}}(\tau)\| \leq \gamma_8(\|(x,\xi)\|)e^{-\alpha\tau} \tag{A.6}$$

for some function $\gamma_8 \in \mathcal{K}_\infty$ and we conclude that the first integral on the right hand side of (A.5) exists. It remains to prove the existence of the integral

$$\int_0^\infty \psi_{\tilde{x}}(\tau) \frac{\partial^2 \tilde{x}}{\partial x_j \partial x_i}(\tau)\, d\tau \tag{A.7}$$

or, using the estimate (2.30), to prove the boundedness for all $\tau \geq 0$ and $0 < a < \alpha$ of

$$\mu(\tau) := \frac{\partial^2 \tilde{x}}{\partial x_j \partial x_i}(\tau)\, e^{-a\tau} \tag{A.8}$$

which satisfies

$$\frac{d\mu}{d\tau} = -a\mu + \left(\frac{\partial f}{\partial x} + \frac{\partial h}{\partial x}\right)\Big|_{(\tilde{x}(\tau),\tilde{\xi}(\tau))} \mu + R(\tau) \tag{A.9}$$

with initial condition $\mu(0) = 0$. Denoting by F_k the k-th column of the matrix $(\frac{\partial f}{\partial x} + \frac{\partial h}{\partial x})\big|_{(\tilde{x}(\tau),\tilde{\xi}(\tau))}$, the k-th component of the vector $R(\tau)$ is given by

$$R_k(\tau) := \left(e^{-\frac{a}{2}} \frac{\partial \tilde{x}}{\partial x_i}(\tau)\right)^T \frac{\partial F_k}{\partial x}\Big|_{(\tilde{x}(\tau),\tilde{\xi}(\tau))} \left(e^{-\frac{a}{2}} \frac{\partial \tilde{x}}{\partial x_j}(\tau)\right) \tag{A.10}$$

By Theorem 2, $R(\tau)$ asymptotically converges to zero. As a consequence, the differential equation (A.9) for μ has the same structure as the differential equation (2.22) for χ, and the rest of the proof of Theorem 2 can be used to conclude that $\mu(\tau)$ converges to zero as $\tau \to \infty$. □

A.3 Proof of Corollary 2

We start with the case when $r = 1$. For an arbitrary initial condition (x, ξ), the assumption (A3') implies

$$\frac{\partial f}{\partial z}\Big|_{\tilde{x}(\tau)} \to F, \text{ as } \tau \to \infty$$

with F a constant matrix with eigenvalues with real parts strictly smaller than α. Now a has to be chosen such that $\max\{Re(\lambda_i(F)),\ i = 1, \ldots, n\} < a < \alpha$. By assumption (A3'), such a constant exists. Then χ and ν satisfy

$$\begin{aligned} \dot{\chi} &= (F - aI)\chi + \mathcal{B}_1\chi \\ \dot{\nu} &= (F - aI)\nu + \mathcal{B}_2\nu + \beta \end{aligned}$$

Because $F - aI$ is Hurwitz, and $\mathcal{B}_i$ and β converge to 0, we conclude that χ and ν converge to 0. The rest of the proof for the case $r = 1$ is identical to the proof of Theorem 2.

To prove that Ψ is twice continuously differentiable when $r = 2$ we consider again $\mu(\tau)$ defined by (A.8) and rewrite its dynamics as

$$\frac{\partial \mu}{\partial \tau} = (F - aI)\mu + \mathcal{B}(\tau) + R(\tau)$$

where $\mathcal{B}(\tau)$ converges to 0 as $\tau \to \infty$. The vector $R(\tau)$, given by (A.10), converges to 0 provided that $0 < a < \alpha$ and $e^{-\frac{1}{2}a\tau}\frac{\partial\tilde{x}}{\partial x}$ is bounded. The latter condition is satisfied if $\frac{1}{2}a > \max\{Re(\lambda_i(F))\}$ and $a < \alpha$. That such an a exists is guaranteed by (A3'), since for $r = 2$, $\frac{1}{2}\alpha > \max\{Re(\lambda_i(F))\}$. Thus, $\mu(\tau)$ is bounded and converges to 0. The existence of second partial derivatives of Ψ follows as in the proof of Corollary 1. The existence of partial derivatives of order higher than 2 when $r > 2$ can be shown in the same way. □

A.4 Proof of Corollary 3

Below "linearization" means "Jacobian linearization at 0". We will show that the linearization of the control law (3.5) is a stabilizing feedback for the linearization of (Σ_1). This implies that the linearization of (Σ_1, u_1) is asymptotically stable and therefore exponentially stable.

First we write the approximations around the origin of relevant functions by keeping the terms of lowest order:

$$\begin{aligned}
W(x) &= x^T W_1 x + h.o.t. \\
\Psi(x,\xi) &= x^T \Psi_1 \xi + \xi^T \Psi_2 \xi + h.o.t. \\
h(x,\xi) &= H_0 \xi + h.o.t. \\
g(x,\xi) &= g_0 + h.o.t \\
f(x) &= Fx + h.o.t.
\end{aligned}$$

where $h.o.t.$ stands for "higher order terms." The linearization of the control law u_1 becomes

$$u_{1l} = -2x^T W_1 g_0 - \xi^T \Psi_1^T g_0 - x^T \Psi_1 b - 2\xi^T \Psi_2 b \tag{A.11}$$

For the linearization of (Σ_1) , we use the same construction as in Theorem 3 and design the linear control law

$$\bar{u} = -2x^T W_1 g_0 - \xi^T \Psi_{l1}^T g_0 - x^T \Psi_{l1} b - 2\xi^T \Psi_{l2} b \tag{A.12}$$

where the matrices Ψ_{l1} and Ψ_{l2} are obtained from

$$\Psi_l(x,\xi) \triangleq \int_0^\infty 2\tilde{x}^T(s) W_1 H_0 \tilde{\xi}(s)\, ds = x^T \Psi_{l1} \xi + \xi^T \Psi_{l2} \xi$$

With the notations of Theorem 3, the control law (A.12) is of the form $\bar{u} = -2\bar{b}^T \bar{P} \zeta$ and, therefore, it stabilizes the linearized system $(\bar{\Sigma}_1)$.

To prove that the control law (A.12) is identical to the linearization (A.11) of the input u_1 we need to show that $\Psi_1 = \Psi_{l1}$ and $\Psi_2 = \Psi_{l2}$. Using the PDE (4.1) or comparing the dominant (quadratic) terms in $\dot{V}_1$, one easily verify that matrices Ψ_1 and Ψ_2 are solutions of

$$\begin{aligned}
F^T \Psi_1 + \Psi_1 A &= -W_1 H_0 \\
A^T \Psi_2 + \Psi_2 A &= H_0^T \Psi_1
\end{aligned} \tag{A.13}$$

Similarly one can verify that Ψ_{l1} and Ψ_{l2} must satisfy the same equations (A.13). The first matrix equation is a Sylvester equation for which the existence and uniqueness of solutions hold provided that the spectra of A and $-F$ are disjoint, i.e. $\lambda_i(A) \neq -\lambda_j(F)$. This condition is always satisfied because it A is Hurwitz and F is Lyapunov stable. The second equation is a Lyapunov equation and the solution always exists and is unique because A is Hurwitz. By uniqueness of solutions of (A.13) we deduce that $\Psi_{l1} = \Psi_1$ and $\Psi_{l2} = \Psi_2$, which ends the proof. $\square$